EXERCISES FOR CHAPTER 1

Exercise 0.1 (A quasiprojective variety with non-finitely generated section ring)**.** Suppose that W is the union of two planes P_1, P_2 in $\mathbb{P}^3$ which meet along a line ℓ. Let ℓ' be a line in P_1 that is different from ℓ (so ℓ and ℓ' meet along a point p) and set $X = W \backslash \ell'$. Prove that $\mathcal{O}_X(X)$ is not finitely generated.

Exercise 0.2 (A quasiprojective variety with non-finitely generated section ring)**.** Let E be an elliptic curve. Let $\mathcal{L}$ be a non-torsion element of $\mathrm{Pic}^0(E)$ and let $\mathcal{M}$ be a line bundle of degree 2. Consider the graded ring $\oplus_{m \geq 0} H^0(E, \mathrm{Sym}^m(\mathcal{L} \oplus \mathcal{M}))$. Prove that this ring is not finitely generated. (Hint: consider the piece coming from $\mathcal{L}^{m-1} \otimes \mathcal{M}$.)

Using the relative Spec construction, find a quasiprojective variety with non-finitely generated section ring.

Exercise 0.3 (A complete non-projective scheme)**.** One way to construct a complete variety that is not projective is to ensure that the Nakai-Moishezon criterion can't possibly hold for any divisor D.

Let Y denote the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point. By composing the blow-down with a projection map, we have a morphism $f : Y \to \mathbb{P}^1$ such that every fiber but one is isomorphic to $\mathbb{P}^1$ and the remaining fiber is the union of two copies of $\mathbb{P}^1$.

We construct a surface X by gluing two copies Y_1, Y_2 of Y together in the following way. Take one component of the reducible fiber in Y_1, and glue it to a smooth fiber in Y_2. Next take one component of the reducible fiber in Y_2, and glue it to a smooth fiber in Y_1. (Note that the two gluing operations are disjoint and so do not interfere with each other. If you are not sure why you can glue along closed subsets, you can look in Vakil's notes or take it for granted.)

Using the invariance of degree in families, explain why it is impossible for a Cartier divisor D on X to have positive degree against the fiber-like curves in Y_1, Y_2. Conclude that X is not projective. (Hironaka's example is a smooth complete threefold which contains this surface inside of it, and thus cannot be projective.)

Exercise 0.4. Find an example of a smooth projective threefold X and a nef divisor D such that

- $D \cdot C > 0$ for every curve C,
- $D^3 > 0$, and
- D is not ample.

Exercise 0.5. Let S be a smooth projective surface and let D be a Cartier divisor.

1. Use (Hirzebruch-)Riemann-Roch to compute the Euler characteristic $\chi(S, \mathcal{O}_X(D))$.
2. Fix an ample divisor H on S. Suppose that $D^2 > 0$ and that $D \cdot H > 0$. Prove that $|mD|$ is non-empty for m sufficiently large. (Hint: use (1). To handle the potential growth of $H^2(S, \mathcal{O}_S(mD))$, appeal to Serre duality.)
3. Conclude that the "positive cone"

$$Q = \{D \in N^1(X)_{\mathbb{R}} | D^2 \geq 0 \text{ and } D \cdot H \geq 0\}$$

is always contained in $\overline{\text{Eff}}_1(S)$. Show that if $Q \subsetneq \overline{\text{Eff}}_1(S)$ then S must carry a curve of negative self-intersection.

For example, Kovács proves that for a K3 surface S we either have $\overline{\text{Eff}}_1(S) = Q$ or $\overline{\text{Eff}}_1(S)$ is the closure of the cone generated by the smooth rational curves on S (which must have negative self-intersection).

Exercise 0.6. Consider the example from class: let E be an elliptic curve without CM and let F_1, F_2, Δ be a basis for $N^1(E \times E)_{\mathbb{Z}}$. Recall that $\alpha = xF_1 + yF_2 + z\Delta$ is contained in $\text{Nef}^1(X)$ if and only if $x + y + z \geq 0$ and $xy + yz + zx \geq 0$.

(1) Recall that $E \times E$ admits a $SL_2(\mathbb{Z})$ action sending $(x, y) \mapsto (ax + cy, bx + dy)$. Verify that the action of this group on $N^1(X)_{\mathbb{R}}$ (in the basis above) is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a^2 - ab & c^2 - cd & (a+c)^2 - (a+c)(b+d) \\ b^2 - ab & d^2 - cd & (b+d)^2 - (a+c)(b+d) \\ ab & cd & (a+c)(b+d) \end{bmatrix}.$$

(2) Compute all the rational rays on the boundary of $\text{Nef}^1(X)$. (Hint: remember how to find the rational points on a circle.) Show that the orbit of F_1 under the $SL_2(\mathbb{Z})$-action will yield precisely the rational rays on the boundary of $\text{Nef}^1(X)$.

Exercise 0.7. Let E be the elliptic curve defined as the quotient of $\mathbb{C}$ by the lattice $\mathbb{Z}[i]$. This curve has an "extra" automorphism $\gamma : E \times E$ given by multiplication by i on $\mathbb{C}$. Thus $\text{Hom}(E, E) \cong \mathbb{Z}^2$. This means that $N^1(E \times E)_{\mathbb{Z}}$ has rank 4; it is generated by F_1, F_2, Δ, Γ where Γ is the graph of the automorphism γ.

(1) Write down the intersection table for the generators of $N^1(E \times E)_{\mathbb{Z}}$.
(2) Compute the nef cone of $E \times E$ – that is, determine when $wF_1 + xF_2 + y\Delta + z\Gamma$ is nef.

Exercise 0.8. Let S be the blow-up of $\mathbb{P}^2$ at two distinct points. (Since the automorphism group of $\mathbb{P}^2$ is 2-transitive it does not matter which two points we pick.) Compute $\overline{\text{Eff}}_1(S)$ and $\text{Nef}^1(S)$.

Exercise 0.9. Let S be the blow-up of $\mathbb{P}^2$ at three distinct points. Up to automorphisms, the only feature of the three points is whether or not they are collinear. Compute $\overline{\text{Eff}}_1(S)$ and $\text{Nef}^1(S)$ when the points are not collinear, and also when they are.

Exercise 0.10. Let S be a blow-up of $\mathbb{P}^2$ at m points in general position, where $2 \leq m \leq 7$. Recall that $\text{Pic}(S)$ is generated by the pullback of the hyperplane class H and the m exceptional divisors E_i.

(1) Prove that every (-1)-curve on S has class E_i, $H - E_i - E_j$ for $i \neq j$, or $2H - E_i - E_j - E_k - E_l - E_m$ for distinct indices. What is the geometric interpretation of each type?
(2) Count the number of (-1)-curves on S. (This is less fun as m gets larger.)

Exercise 0.11 (Taken from a note by Christian Schnell)**.** Fix a pencil of elliptic curves in $\mathbb{P}^2$ such that every member of the pencil is irreducible. By blowing up the 9 basepoints of the pencil we obtain a birational model $\phi : X \to \mathbb{P}^2$ equipped with a morphism $\pi : X \to \mathbb{P}^1$ whose fibers are the elliptic curves in our pencil.

(1) Show that $N^1(X)_{\mathbb{Z}}$ has rank 10 and is generated by the pullback H of the hyperplane class on $\mathbb{P}^2$ and the 9 exceptional curves $E_1, \ldots, E_9$. Show that the canonical divisor of X is $-3H + \sum_{i=1}^{9} E_i$.

(2) Show that a curve $C \subset X$ is a (-1)-curve if and only if it has class $bH - \sum a_i E_i$ where

$$\sum_{i=1}^{9} a_i = 3b - 1 \qquad\qquad \sum_{i=1}^{9} a_i^2 = b^2 + 1$$

In particular show that every (-1)-curve is a section of π.

(3) We have a group action on the sections of π in the following way. Suppose C_1, C_2 are sections of π. For any smooth fiber F of π, we can apply the group law of F to the points $(C_1 \cap F)$ and $(C_2 \cap F)$ to get a new point of F. We can apply this operation to every smooth fiber simultaneously and take a closure to obtain a section $C_1 * C_2 = C_3$. In this way we can obtain infinitely many (-1)-curves on $\mathbb{P}^2$.

Here is an explicit formula. Suppose that C is a (-1)-curve with class $bH - \sum a_i E_i$. Assume for simplicity that C is not any of the E_j. Prove that $C' = C * E_j$ has class determined by the formulas

$$a_i' = b - a_i - a_j + \delta_{ij} \qquad\qquad b' = 2b - 3a_j + 1$$

(Hint: for $i \neq j$, show that C' and E_i will intersect along a fiber F if and only if $\phi(C' \cap F)$, $\phi(E_i)$ and $\phi(E_j)$ are collinear in $\mathbb{P}^2$.)