Radical Expressions

Simplifying Radical Expressions

Definition: A *radical expression* is any expression of the form $\sqrt[n]{\prod_{i=1}^{n} x_i^k}$, where n and k are any positive integers and in "Basic algebraic notation", the $x_1, x_2, x_3, ...$ are represented as $x, y, z...$

Before we look at a few examples, we need to be comfortable with the familiar integral powers of the first few natural numbers, $n \in \mathbb{N}$. That is:

$n = 2$: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, ...$ (i.e. the perfect squares)

$n = 3$: $1, 8, 27, 64, 125, 625, ...$ (i.e. the perfect cubes)

$n = 4$: $1, 16, 81, 256, ...$ (i.e. the perfect fourth powers)

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We should memorize these perfect powers for the first few integers as we will use them in the examples below. Notice that we listed less of them as n gets larger. This is primarily because we do not recommend you memorizing (or at least becoming familiar with) the others as they are not used very often because powers of integers grow large fast and we also rarely work with radical expressions where $n > 4$ (as a matter of fact, usually n is 2 for computing "Square roots" and occasionally n is 3 for computing "Cube roots." However, fourth and fifth roots occur less often and if such a case were to arise, you can count on the power of n needed in order to simplify the radical expression to be relatively small!

We will begin our explanation of simplifying such expressions in the case where $n = 2$, so that the"Rradical expression" is really just a "Square Root." The reason for this is two-fold. First, we must understand the most basic case before moving on to the more general case (i.e. n = 3,4,5,....) and more importantly because, in most cases when one sais ,"Radical 2," they mean, "Square root of 2." *That is, if no n is specified, it is understood to be 2.*

Let's look at some examples of simplifying radical expressions where the radicand is simply a number and not an expression containing several (powers of) variables.

Example: Simplify $\sqrt{20}$

Solution:

$$\sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$$

(Note that 4 is the largest perfect square that divides 20 (with zero remainder!)

QED

Now, let's explore the case where the radicand possibly contains numbers and variables.

Example: Simplify $\sqrt{72x^5y^3}$. Assume all variables represent positive real numbers.

Solution:

Since 36 is the largest perfect square that divides 72, x^4 is a perfect square [highest power divisible by 2 (since square root)], and y^2 is a perfect square, we obtain:

$$\sqrt{72x^5y^3} = \sqrt{36 \cdot 2 \cdot x^4 \cdot x \cdot y^2 y} = 6x^2 y\sqrt{xy}$$

QED

Now that we have practice with square roots, we will look at radicals that contain higher roots (i.e. indices) than two. Just remember, if no index is given in the radical, it is assumed to be two (i.e. a square root).

Example: Simplify $\sqrt[3]{16x^5y^4}$. Assume that all variables represent positive real numbers.

Solution: Since 8 is the largest perfect cube that divides 16, 3 is the largest integer less than 5 that is divisible by 3 (i.e. cube root), and 3 is the largest integer less than 4, we obtain,

$$\sqrt[3]{16x^5y^4} = \sqrt[3]{8 \cdot 2 \cdot x^3 \cdot x^2 \cdot y^3 \cdot y} = 2xy\sqrt[3]{2x^2y}$$

QED

Following the above reasoning, you can see that *your answer is not in the most simplified form if each exponent inside the radical is not less than the index of the radical.*

Example: Simplify $\sqrt[3]{108x^6z^5}$. Assume that all variables represent positive real numbers.

Solution: We can see that 27 is the largest perfect cube that divides 108, six was divisible by 3 (so we did not rewrite it), and we rewrite z^5 as $z^3 \cdot z^2$ since 3 is the largest integer less than 5.

$$\sqrt[3]{108x^6z^5} = \sqrt[3]{27 \cdot 4 \cdot x^6 \cdot z^3 \cdot z^2} = 3x^2y\sqrt[3]{z^2}$$

QED

Example: Simplify $\sqrt[4]{48x^6z^9}$. Assume that all variables represent positive real numbers.

Solution: Using similar reasoning as the above examples but with $n = 4$, we obtain:

$$\sqrt[4]{48x^6z^9} = \sqrt[4]{16 \cdot 3 \cdot x^4 \cdot x^2 \cdot z^8 \cdot z} = 2xz^2\sqrt[4]{3x^2z} .$$

QED

<u>Operations on Radical expressions (MA.912.A.6.2)</u>

<u>Addition and Subtraction of Radical Expressions</u>

Addition and subtraction of radical expressions (like any expression) requires that the radicals be like terms. For example, just like 3x + 2x = 5x, we can see that $3\sqrt{6} + 2\sqrt{6} = 5\sqrt{6}$ if x = $\sqrt{6}$. However, if the radicand does not appear to be a like term, it may indeed be a like term. That is, before we add and subtract any radical expressions, we must simplify each term (i.e. radical expression) as we did in the previous examples. That is they may be like terms after we simplify each radical.

Example: Add: $3\sqrt{24} + \sqrt{54}$

Solution:

$$\begin{aligned} 3\sqrt{24} + \sqrt{54} &= \\ 3\sqrt{4 \cdot 6} + \sqrt{9 \cdot 6} &= \\ 6\sqrt{6} + 3\sqrt{6} &= \\ &= 9\sqrt{6} \end{aligned}$$

QED

Example: Subtract: $4\sqrt{18}-2\sqrt{8}$

Solution:

$$\begin{aligned} 4\sqrt{18}-2\sqrt{8} &= \\ 4\sqrt{9\cdot 2}-2\sqrt{4\cdot 2} &= \\ 12\sqrt{2}-4\sqrt{2} &= \\ &= 8\sqrt{2} \end{aligned}$$

QED

Example: Simplify**.** Assume all variable represent positive real numbers**:**

$$3\sqrt{72x^3}+5\sqrt{32x^3}-7\sqrt{18x^3}$$

Solution:

$$\begin{aligned} &3\sqrt{72x^3}+5\sqrt{32x^3}-7\sqrt{18x^3} \\ &= 3\sqrt{36\cdot 2\cdot x^2\cdot x}+5\sqrt{16\cdot 2\cdot x^2\cdot x}-7\sqrt{9\cdot 2\cdot x^2\cdot x} \\ &= 18x\sqrt{2x}+20x\sqrt{2x}-21x\sqrt{2x} \\ &= 17x\sqrt{2x} \end{aligned}$$

QED

Multiplication and division of radical expressions

In order to multiply radical expressions, we simply multiply all of the factors outside the radical and leave them outside the radical in your answer and multiply all of the factors inside the radical and leave them inside the radical in your answer.

Example: Multiply: $2\sqrt{6}\cdot 3\sqrt{3}$

Solution:

$$\begin{aligned} 2\sqrt{6}\cdot 3\sqrt{3} &= 6\sqrt{18} \\ &= 6\sqrt{9\cdot 2} = 18\sqrt{2} \end{aligned}$$

QED

Notice that in the above example, we multiplied the factors inside the radical and left that product inside the radical and we multiplied the factors outside the radical and left that product outside the radical. Lastly, we simplified the radical to its simplest form (i.e. that is there are no perfect square factors inside the square root!).

Example: Divide: $\dfrac{8\sqrt{10}}{4\sqrt{5}}$

Solution: $\dfrac{8\sqrt{10}}{4\sqrt{5}} = 2\sqrt{2}$ Divide inside and outside the radical independently

QED

When dividing radicals, sometimes we may end up with a radical in the denominator. In order for your answer to be in simplified form, *we are not allowed to leave radicals in the denominator! (*almost similar to the idea that we do not leave negatives in the denominator, but a little trickier to rectify).

We can manipulate any expression that has a radical in the denominator to an equivalent expression that has no radical in the denominator by using a technique called rationalizing the denominator. Let's explore this technique a bit.

<u>Rationalizing the denominator</u>

In order to rationalize the denominator, we multiply the radical expression by 1, but in a special form.

Example: Simplify $\dfrac{2}{\sqrt{3}}$ (Note: The directions may instead say "Divide: $\dfrac{2}{\sqrt{3}}$")

Solution: $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}(1) = \dfrac{2}{\sqrt{3}}\left(\dfrac{\sqrt{3}}{\sqrt{3}}\right) = \dfrac{2\sqrt{3}}{3}$ (Recall, if x is positive, $\sqrt{x} \cdot \sqrt{x} = x$)

QED

Example:

Simplify: $\dfrac{3}{2-\sqrt{5}}$

Solution: Here we use the conjugate of the denominator to multiply the expression by the special form of 1. That is,

$$
\begin{aligned}
\frac{3}{2-\sqrt{5}} &= \frac{3}{2-\sqrt{5}}(1) \\
&= \left(\frac{3}{2-\sqrt{5}}\right)\left(\frac{2+\sqrt{5}}{2+\sqrt{5}}\right) \\
&= \frac{6+3\sqrt{5}}{-1} = -6-3\sqrt{5}
\end{aligned}
$$

QED

Sometimes it is more convenient to simplify the radical before performing the multiplication or division and sometimes it is more convenient to simplify the radical after performing your operation. Usually simplifying the radicals before multiplying and dividing is more efficient since we will not have to simplify as much later, but there are times when simplifying after is better.

For example, if asked you to simplify $\frac{\sqrt{75}}{\sqrt{25}}$, clearly it is better to perform the operation first as we simply obtain $\frac{\sqrt{75}}{\sqrt{25}} = \sqrt{\frac{75}{25}} = \sqrt{3}$. On the contrary, suppose we were asked to simplify $\frac{\sqrt{27}}{5\sqrt{3}}$, then clearly simplifying the radicals before performing the division may be more efficient, as demonstrated in the next example. However, we can proceed either way and get the same result.

Example:

Divide: $\frac{\sqrt{27}}{5\sqrt{3}}$

Solution: Let's look at two different methods in order to demonstrate the idea discussed above.

<u>Method 1 (Simplifying first)</u>

$$
\frac{\sqrt{27}}{5\sqrt{3}} = \frac{1}{5}\sqrt{\frac{27}{3}} = \frac{1}{5}\sqrt{9} = \frac{3}{5}
$$

Method 2 (Simplifying last)

$$\frac{\sqrt{27}}{5\sqrt{3}} = \frac{\sqrt{9 \cdot 3}}{5\sqrt{3}} = \frac{3\sqrt{3}}{5\sqrt{3}} = \frac{3}{5}$$

QED

Example:

Multiply: $\left(5-\sqrt{3}\right)\left(\sqrt{7}+\sqrt{2}\right)$

Solution:

$$\left(5-\sqrt{3}\right)\left(\sqrt{7}+\sqrt{2}\right) = 5\sqrt{7}+5\sqrt{2}-3\sqrt{7}-\sqrt{6} \qquad \text{FOIL}$$

$$=2\sqrt{7}+5\sqrt{2}-\sqrt{6}$$

QED

Example:

Multiply: $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)$

Solution: $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right) = 4-5 = -1$ Difference of two (perfect) squares

QED

Example:

Simplify: $\left(3+\sqrt{2}\right)^2$

Solution:

$$\left(3+\sqrt{2}\right)^2$$

$$= 3^2 + 2 \cdot 3\sqrt{2} + \left(\sqrt{2}\right)^2$$

$$= 9 + 6\sqrt{2} + 2$$

$$= 11 + 6\sqrt{2}$$

QED

Example:

Simplify: $\sqrt[3]{16x^2} \cdot \sqrt[3]{54x^3}$

Solution:

$$\begin{aligned}
\sqrt[3]{16x^2} \cdot \sqrt[3]{54x^3} &= \sqrt[3]{8 \cdot 2x^2} \cdot \sqrt[3]{27 \cdot 2x^3} \\
&= 2\sqrt[3]{2x^2} \cdot 2\sqrt[3]{2x^3} = 4\sqrt[3]{4x^5} \\
&= 4\sqrt[3]{4x^3 \cdot x^2} = 4x\sqrt[3]{4x^2}
\end{aligned}$$

QED

Example:

Simplify. Assume all variables represent positive real numbers: $\dfrac{\sqrt[3]{81x^5}}{\sqrt[3]{8x}}$

Solution:

$$\frac{\sqrt[3]{81x^5}}{\sqrt[3]{8x}} = \frac{\sqrt[3]{27 \cdot 3 \cdot x^3 \cdot x^2}}{2\sqrt[3]{x}} = \frac{3x\sqrt[3]{3x^2}}{2\sqrt[3]{x}}$$

$$= \left(\frac{3x\sqrt[3]{3x^2}}{2\sqrt[3]{x}}\right)(1) = \left(\frac{3x\sqrt[3]{3x^2}}{2\sqrt[3]{x}}\right)\left(\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}}\right) \qquad \text{Rationalize the denominator}$$

$$= \frac{3x\sqrt[3]{3x^4}}{2\sqrt[3]{x^3}} = \frac{3x\sqrt[3]{3x^4}}{2x} = \frac{3x^2\sqrt[3]{3x}}{2x}$$

QED