Handout 5
The Reciprocal Lattice

In this lecture you will learn:

• Fourier transforms of lattices
• The reciprocal lattice
• Brillouin Zones
• X-ray diffraction
• Fourier transforms of lattice periodic functions

Fourier Transform (FT) of a 1D Lattice

Consider a 1D Bravais lattice:

\[ \mathbf{a}_1 = a \hat{x} \]

Now consider a function consisting of a "lattice" of delta functions – in which a delta function is placed at each lattice point:

\[ f(x) = \sum_{n=-\infty}^{\infty} \delta(x - na) \]

The FT of this function is (as you found in your homework):

\[ f(k_x) = \int_{-\infty}^{\infty} dx \sum_{n=-\infty}^{\infty} \delta(x - na) e^{-i k_x x} = \sum_{n=-\infty}^{\infty} e^{i k_x na} = 2\pi \sum_{m=-\infty}^{\infty} \delta(k_x - m \frac{2\pi}{a}) \]

The FT of a train of delta functions is also a train of delta functions in k-space
The reciprocal lattice is defined by the position of the delta-functions in the FT of the actual lattice (also called the direct lattice).

**Direct lattice (or the actual lattice):**

$$\mathbf{a}_1 = a \mathbf{x}$$

**Reciprocal lattice:**

$$\mathbf{b}_1 = \frac{2\pi}{a} \mathbf{\hat{x}}$$

For the 1D Bravais lattice, the position vector $\mathbf{R}_n$ of any lattice point is given by:

$$\mathbf{R}_n = n \mathbf{a}_1$$

The FT of this function is:

$$f(k_x) = \int_{-\infty}^{\infty} dx \sum_{n=-\infty}^{\infty} \delta(\mathbf{x} - \mathbf{R}_n) e^{-i k_x \mathbf{x}} = \sum_{n=-\infty}^{\infty} e^{i k \cdot \mathbf{R}_n}$$

The reciprocal lattice in k-space is defined by the set of all points for which the k-vector satisfies,

$$e^{i k \cdot \mathbf{R}_n} = 1$$

for ALL $\mathbf{R}_n$ of the direct lattice.

For the points in k-space belonging to the reciprocal lattice the summation $\sum_{n=-\infty}^{\infty} e^{i k \cdot \mathbf{R}_n}$ becomes very large!
Reciprocal Lattice of a 1D Lattice

For the 1D Bravais lattice,\[ \mathbf{a}_1 = a \hat{x} \]
The position vector \( \mathbf{R}_n \) of any lattice point is given by: \[ \mathbf{R}_n = n \mathbf{a}_1 \]

The reciprocal lattice in \( k \)-space is defined by the set of all points for which the \( k \)-vector satisfies, \[ e^{i \mathbf{k} \cdot \mathbf{R}_n} = 1 \]
for all \( \mathbf{R}_n \) of the direct lattice

For \( \mathbf{k} \) to satisfy \( e^{i \mathbf{k} \cdot \mathbf{R}_n} = 1 \), it must be that for all \( \mathbf{R}_n \): \[ \mathbf{k} \cdot \mathbf{R}_n = 2\pi \times \{ \text{integer} \} \]
\[ \Rightarrow k_x n a = 2\pi \times \{ \text{integer} \} \]
\[ \Rightarrow k_x = m \frac{2\pi}{a} \quad \{ \text{where } m \text{ is any integer} \} \]

Therefore, the reciprocal lattice is:

Reciprocal Lattice of a 2D Lattice

Consider the 2D rectangular Bravais lattice:

If we place a 2D delta function at each lattice point we get the function:
\[ f(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - n a) \delta(y - m c) \]

The above notation is too cumbersome, so we write it in a simpler way as:
\[ f(\mathbf{r}) = \sum_j \delta^2(\mathbf{r} - \mathbf{R}_j) \]

The summation over "\( j \)" is over all the lattice points

A 2D delta function has the property: \( \{ d^2\mathbf{r} \quad \delta^2(\mathbf{r} - \mathbf{r}_0) g(\mathbf{r}) = g(\mathbf{r}_0) \) and it is just a product of two 1D delta functions corresponding to the \( x \) and \( y \) components of the vectors in its arguments: \( \delta^2(\mathbf{r} - \mathbf{r}_0) = \delta(x - r_0 x) \delta(y - r_0 y) \)

Now we Fourier transform the function \( f(\mathbf{r}) \):
\[ f(\mathbf{k}) = \int d^2\mathbf{r} \quad f(\mathbf{r}) \quad e^{-i \mathbf{k} \cdot \mathbf{r}} = \int d^2\mathbf{r} \quad \sum_j \delta^2(\mathbf{r} - \mathbf{R}_j) \quad e^{-i \mathbf{k} \cdot \mathbf{r}} \]
\[ = \sum_j e^{-i \mathbf{k} \cdot \mathbf{R}_j} = \frac{(2\pi)^2}{ac} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(k_x n \frac{2\pi}{a}) \delta(k_y m \frac{2\pi}{c}) \]
Reciprocal Lattice of a 2D Lattice

\[ f(\mathbf{k}) = \sum_{j} e^{-i \mathbf{k} \cdot \mathbf{R}_j} = \frac{(2\pi)^2}{ac} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(k_x - n \frac{2\pi}{a}) \delta(k_y - m \frac{2\pi}{c}) \]

- Note also that the reciprocal lattice in \( \mathbf{k} \)-space is defined by the set of all points for which the \( \mathbf{k} \)-vector satisfies,
  \[ e^{i \mathbf{k} \cdot \mathbf{R}_j} = 1 \]
  for all \( \mathbf{R}_j \) of the direct lattice

- Reciprocal lattice as the FT of the direct lattice or as set of all points in \( \mathbf{k} \)-space for which, \( \exp(i \mathbf{k} \cdot \mathbf{R}_j) = 1 \) for all \( \mathbf{R}_j \), are equivalent statements

Reciprocal Lattice of a 2D Lattice

- The reciprocal lattice of a Bravais lattice is always a Bravais lattice and has its own primitive lattice vectors, for example, \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) in the above figure

- The position vector \( \mathbf{G} \) of any point in the reciprocal lattice can be expressed in terms of the primitive lattice vectors:
  \[ \mathbf{G} = n \mathbf{b}_1 + m \mathbf{b}_2 \]

So we can write the FT in a better way as:

\[ f(\mathbf{k}) = \frac{(2\pi)^2}{\Omega_2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(k_x - n \frac{2\pi}{a}) \delta(k_y - m \frac{2\pi}{c}) = \frac{(2\pi)^2}{\Omega_2} \sum_j \delta^2(\mathbf{k} - \mathbf{G}_j) \]

where \( \Omega_2 = ac \) is the area of the direct lattice primitive cell
Reciprocal Lattice of a 3D Lattice

Consider an orthorhombic direct lattice:

\[ \vec{R} = n \vec{a}_1 + m \vec{a}_2 + p \vec{a}_3 \quad \text{where } n, m, \text{ and } p \text{ are integers} \]

Then the corresponding delta-function lattice is:

\[ f(\vec{r}) = \sum_j \delta^3(\vec{r} - \vec{R}_j) \]

A 3D delta function has the property:

\[ \int d^3 \vec{r} \delta^3(\vec{r} - \vec{r}_0) g(\vec{r}) = g(\vec{r}_0) \]

The reciprocal lattice in k-space is defined by the set of all points for which the k-vector satisfies:

\[ \exp(i \vec{k} \cdot \vec{R}_j) = 1 \quad \text{for all } \vec{R}_j \text{ of the direct lattice} \]

The above relation will hold if \( \vec{k} \) equals \( \vec{G} \):

\[ \vec{G} = n \vec{b}_1 + m \vec{b}_2 + p \vec{b}_3 \quad \text{and} \quad \vec{b}_1 = \frac{2\pi}{a} \hat{x} \quad \vec{b}_2 = \frac{2\pi}{c} \hat{y} \quad \vec{b}_3 = \frac{2\pi}{d} \hat{z} \]

Finally, the FT of the direct lattice is:

\[ f(\vec{k}) = \int d^3 \vec{r} f(\vec{r}) e^{-i \vec{k} \cdot \vec{r}} = \int d^3 \vec{r} \sum_j \delta^3(\vec{r} - \vec{R}_j) e^{-i \vec{k} \cdot \vec{r}} \]

\[ = \sum_j e^{-i \vec{k} \cdot \vec{R}_j} = \left( \frac{2\pi}{acd} \right)^3 \sum_j \delta^3(\vec{k} - \vec{G}_j) = \left( \frac{2\pi}{\Omega_3} \right)^3 \sum_j \delta^3(\vec{k} - \vec{G}_j) \]

Direct Lattice Vectors and Reciprocal Lattice Vectors

\[ \vec{R} = n \vec{a}_1 + m \vec{a}_2 \]

\[ \vec{G} = n \vec{b}_1 + m \vec{b}_2 \]

Remember that the reciprocal lattice in k-space is defined by the set of all points for which the k-vector satisfies,

\[ e^{i \vec{k} \cdot \vec{R}} = 1 \quad \text{for all } \vec{R} \text{ of the direct lattice} \]

So for all direct lattice vectors and all reciprocal lattice vectors we must have:

\[ e^{i \vec{G} \cdot \vec{R}} = 1 \]
Reciprocal Lattice of General Lattices in 1D, 2D, 3D

More often than not, the direct lattice primitive vectors, \( \mathbf{a}_1, \mathbf{a}_2, \) and \( \mathbf{a}_3 \), are not orthogonal.

**Question:** How does one find the reciprocal lattice vectors in the general case?

### 1D lattice:
If the direct lattice primitive vector is: \( \mathbf{a}_1 = a \hat{x} \)
and length of primitive cell is: \( \Omega_1 = a \)
Then the reciprocal lattice primitive vector is:

\[
\mathbf{b}_1 = \frac{2\pi}{a} \hat{x}
\]

Note:
\[
\mathbf{a}_1 \cdot \mathbf{b}_1 = 2\pi, \quad e^{i \mathbf{a}_1 \cdot \mathbf{R}_m} = 1
\]

### 2D lattice:
If the direct lattice is in the x-y plane and the primitive vectors are: \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \)
and area of primitive cell is: \( \Omega_2 = |\mathbf{a}_1 \times \mathbf{a}_2| \)
Then the reciprocal lattice primitive vectors are:

\[
\mathbf{b}_1 = \frac{2\pi}{\Omega_2} \mathbf{a}_2 \times \hat{z} \quad \mathbf{b}_2 = \frac{2\pi}{\Omega_2} \mathbf{a}_1 \times \hat{z}
\]

Note:
\[
\mathbf{a}_j \cdot \mathbf{b}_k = 2\pi \delta_{jk}, \quad e^{i \mathbf{a}_j \cdot \mathbf{R}_m} = 1
\]

### 3D lattice:
If the direct lattice primitive vectors are: \( \mathbf{a}_1, \mathbf{a}_2, \) and \( \mathbf{a}_3 \)
and volume of primitive cell is: \( \Omega_3 = |\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)| \)
Then the reciprocal lattice primitive vectors are:

\[
\mathbf{b}_1 = \frac{2\pi}{\Omega_3} \mathbf{a}_2 \times \mathbf{a}_3 \quad \mathbf{b}_2 = \frac{2\pi}{\Omega_3} \mathbf{a}_3 \times \mathbf{a}_1 \quad \mathbf{b}_3 = \frac{2\pi}{\Omega_3} \mathbf{a}_1 \times \mathbf{a}_2
\]

Note:
\[
\mathbf{a}_j \cdot \mathbf{b}_k = 2\pi \delta_{jk}, \quad e^{i \mathbf{a}_j \cdot \mathbf{R}_m} = 1
\]

**Example 2D lattice:**
- \( \mathbf{a}_1 = b \hat{x} \)
- \( \mathbf{a}_2 = \frac{b}{2} \hat{x} + \frac{b}{2} \hat{y} \)
- \( \Omega_2 = |\mathbf{a}_1 \times \mathbf{a}_2| = \frac{b^2}{2} \)

Note:
\[
\mathbf{a}_1 \cdot \mathbf{b}_1 = \frac{2\pi}{b}, \quad e^{i \mathbf{a}_1 \cdot \mathbf{R}_m} = 1
\]

### Example 3D lattice:
- \( \mathbf{a}_1 = \frac{4\pi}{b} \hat{x} + \frac{4\pi}{b} \hat{y} \)
- \( \mathbf{a}_2 = \frac{4\pi}{b} \hat{y} \)
- \( \Omega_2 = |\mathbf{a}_1 \times \mathbf{a}_2| = \frac{4\pi^2}{b^2} \)

Note:
\[
\mathbf{a}_1 \cdot \mathbf{b}_1 = \frac{2\pi}{b}, \quad e^{i \mathbf{a}_1 \cdot \mathbf{R}_m} = 1
\]
The Brillouin Zone

The Wigner-Seitz primitive cell of the reciprocal lattice centered at the origin is called the Brillouin zone (or the first Brillouin zone or FBZ).

1D direct lattice:

Reciprocal lattice:

2D lattice:

Volume/Area/Length of the first Brillouin zone:
The volume (3D), area (2D), length (1D) of the first Brillouin zone is given in the same way as the corresponding expressions for the primitive cell of a direct lattice:

1D \( \Pi_1 = |b_1| \)

2D \( \Pi_2 = |\mathbf{b}_1 \times \mathbf{b}_2| \)

3D \( \Pi_3 = |\mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3)| \)

Note that in all dimensions \( d \) the following relationship holds between the volumes, areas, lengths of the direct and reciprocal lattice primitive cells:

\[ \Pi_d = \left( \frac{2\pi}{\Omega_d} \right)^d \]
Direct Lattice Planes and Reciprocal Lattice Vectors

There is an intimate relationship between reciprocal lattice vectors and planes of points in the direct lattice captured by this theorem and its converse

**Theorem:**
If there is a family of parallel lattice planes separated by distance “d” and \( \hat{n} \) is a unit vector normal to the planes then the vector given by,

\[
\mathbf{G} = \frac{2\pi}{d} \hat{n}
\]

is a reciprocal lattice vector and so is:

\[
m \frac{2\pi}{d} \hat{n}
\]

\( m = \text{integer} \)

**Converse:**
If \( \mathbf{G}_i \) is any reciprocal lattice vector, and \( \mathbf{G} \) is the reciprocal lattice vector of the smallest magnitude parallel to \( \mathbf{G}_i \), then there exist a family of lattice planes perpendicular to \( \mathbf{G}_i \) and \( \mathbf{G} \), and separated by distance “d” where:

\[
d = \frac{2\pi}{|\mathbf{G}|}
\]

**Example: Direct Lattice Planes and Reciprocal Lattice Vectors**

Consider:

\[
\mathbf{G} = \hat{b}_1 + \hat{b}_2 = 2\pi \left( \frac{\hat{x}}{a} + \frac{\hat{y}}{c} \right)
\]

There must be a family of lattice planes normal to \( \mathbf{G} \) and separated by:

\[
\frac{2\pi}{|\mathbf{G}|} = \frac{ac}{\sqrt{a^2 + c^2}}
\]

Now consider:

\[
\mathbf{G} = 2\hat{b}_1 + \hat{b}_2 = 2\pi \left( \frac{2\hat{x}}{a} + \frac{\hat{y}}{c} \right)
\]

There must be a family of lattice planes normal to \( \mathbf{G} \) and separated by:

\[
\frac{2\pi}{|\mathbf{G}|} = \frac{ac}{\sqrt{a^2 + 4c^2}}
\]
The BCC Direct Lattice

Direct lattice: BCC

Reciprocal lattice: FCC

The direct and the reciprocal lattices are not necessarily always the same!

The FCC Direct Lattice

Direct lattice: FCC

Reciprocal lattice: BCC

First Brillouin zone of the BCC reciprocal lattice for an FCC direct lattice
The relationship between delta-functions on a “d” dimensional lattice and its Fourier transform is:

\[ f(\mathbf{r}) = \sum_j \delta^d(\mathbf{r} - \mathbf{R}_j) \quad \leftrightarrow \quad f(\mathbf{k}) = \frac{(2\pi)^d}{\Omega_d} \sum_j \delta^d(\mathbf{k} - \mathbf{G}_j) \]

Suppose \( W(\mathbf{r}) \) is a periodic function with the periodicity of the direct lattice then by definition:

\[ W(\mathbf{r} + \mathbf{R}_j) = W(\mathbf{r}) \]

for all \( \mathbf{R}_j \) of the direct lattice.

One can always write a periodic function as a convolution of its value in the primitive cell and a lattice of delta functions, as shown for 1D below:

Mathematically:

\[ W(x) = W_\Omega(x) \otimes \sum_{n=-\infty}^{\infty} \delta(x - n a) \]

And more generally in “d” dimensions for a lattice periodic function \( W(\mathbf{r}) \) we have:

\[ W(\mathbf{r}) = W_\Omega(\mathbf{r}) \otimes \sum_j \delta^d(\mathbf{r} - \mathbf{R}_j) \]

Value of the function in one primitive cell

Lattice of delta functions
The Reciprocal Lattice and FTs of Periodic Functions

For a periodic function we have:
\[ W(\mathbf{r}) = W_\Omega(\mathbf{r}) \otimes \sum_j \delta^d(\mathbf{r} - \mathbf{R}_j) \]

Its FT is now easy given that we know the FT of a lattice of delta functions:
\[ f(\mathbf{r}) = \sum_j \delta^d(\mathbf{r} - \mathbf{R}_j) \iff \hat{f}(\mathbf{k}) = \frac{(2\pi)^d}{\Omega_d} \sum_j \delta^d(\mathbf{k} - \mathbf{G}_j) \]

We get:
\[ W(\mathbf{k}) = W_\Omega(\mathbf{k}) \times \frac{(2\pi)^d}{\Omega_d} \sum_j \delta^d(\mathbf{k} - \mathbf{G}_j) = \frac{(2\pi)^d}{\Omega_d} \sum_j \delta^d(\mathbf{k} - \mathbf{G}_j) W_\Omega(\mathbf{G}_j) \]

If we now take the inverse FT we get:
\[ W(\mathbf{r}) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} W(\mathbf{k}) e^{i \mathbf{k} \cdot \mathbf{r}} = \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(2\pi)^d}{\Omega_d} \sum_j \delta^d(\mathbf{k} - \mathbf{G}_j) W_\Omega(\mathbf{G}_j) e^{i \mathbf{k} \cdot \mathbf{r}} \]
\[ = \sum_j \frac{W_\Omega(\mathbf{G}_j)}{\Omega_d} e^{i \mathbf{G}_j \cdot \mathbf{r}} \]

The FT looks like reciprocal lattice of delta-functions with unequal weights

A lattice periodic function can always be written as a Fourier series that only has wavevectors belonging to the reciprocal lattice.

The Reciprocal Lattice and X-Ray Diffraction

X-ray diffraction is the most commonly used method to study crystal structures

In this scheme, X-rays of wavevector \( \mathbf{k} \) are sent into a crystal, and the scattered X-rays in the direction of a different wavevector, say \( \mathbf{k}' \), are measured.

If the position dependent dielectric constant of the medium is given by \( \varepsilon(\mathbf{r}) \) then the diffraction theory tells us that the amplitude of the scattered X-rays in the direction of \( \mathbf{k}' \) is proportional to the integral:
\[ S(\mathbf{k} \rightarrow \mathbf{k}') \propto \int d^3 \mathbf{r} \ e^{-i \mathbf{k}' \cdot \mathbf{r}} \varepsilon(\mathbf{r}) e^{i \mathbf{k} \cdot \mathbf{r}} \]

For X-ray frequencies, the dielectric constant is a periodic function with the periodicity of the lattice. Therefore, one can write:
\[ \varepsilon(\mathbf{r}) = \sum_j \varepsilon(\mathbf{G}_j) e^{i \mathbf{G}_j \cdot \mathbf{r}} \]

Plug this into the integral above to get:
\[ S(\mathbf{k} \rightarrow \mathbf{k}') \propto \sum_j \varepsilon(\mathbf{G}_j) (2\pi)^3 \delta(\mathbf{k} + \mathbf{G}_j - \mathbf{k}') \]
\[ \Rightarrow X-rays \ will \ scatter \ in \ only \ those \ directions \ for \ which: \]
\[ \mathbf{k}' = \mathbf{k} + \mathbf{G} \quad \text{where } \mathbf{G} \text{ is some reciprocal lattice vector} \]
\[ \text{Or: } \mathbf{k}' = \mathbf{k} \pm \mathbf{G} \quad \text{Because } -\mathbf{G} \text{ is also a reciprocal vector whenever } \mathbf{G} \text{ is a reciprocal vector} \]

ECE 437 – Spring 2008 – Farhan Rana – Cornell University
The Reciprocal Lattice and X-Ray Diffraction

X-rays will scatter in only those directions for which:

\[ \mathbf{k}' = \mathbf{k} \pm \mathbf{G} \quad (1) \]

Also, the frequency of the incident and diffracted X-rays is the same so:

\[ \omega' = \omega \]

\[ \Rightarrow |\mathbf{k}'| = |\mathbf{k}| \]

\[ \Rightarrow \frac{\mathbf{c}}{|\mathbf{c}|} = \frac{\mathbf{k}}{|\mathbf{k}|} \]

(1) gives:

\[ |\mathbf{k}'|^2 = |\mathbf{k}|^2 + |\mathbf{G}|^2 \pm 2 \mathbf{k} \cdot \mathbf{G} \]

\[ \Rightarrow |\mathbf{k}'|^2 = |\mathbf{k}|^2 + |\mathbf{G}|^2 \pm 2 k \cdot G \]

\[ \Rightarrow |\mathbf{k}'|^2 = |\mathbf{k}|^2 + |\mathbf{G}|^2 \pm 2 k \cdot G \]

\[ \Rightarrow \pm k \cdot G = \frac{|\mathbf{G}|^2}{2} \]

Condition for X-ray diffraction

The condition, \( \mathbf{k} \cdot \mathbf{G} = \pm \frac{|\mathbf{G}|^2}{2} \)

is called the Bragg condition for diffraction.

Incident X-rays will diffract efficiently provided the incident wavevector satisfies the Bragg condition for some reciprocal lattice vector \( \mathbf{G} \).

A graphical way to see the Bragg condition is that the incident wavevector lies on a plane in k-space (called the Bragg plane) that is the perpendicular bisector of some reciprocal lattice vector \( \mathbf{G} \).
The Reciprocal Lattice and X-Ray Diffraction

The condition,
\[ \mathbf{k} \cdot \mathbf{G} = \pm \frac{|G|^2}{2} \]
can also be interpreted the following way:

Incident X-rays will diffract efficiently when the reflected waves from successive atomic planes add in phase

**Recall that there are always a family of lattice planes in real space perpendicular to any reciprocal lattice vector

Condition for in-phase reflection from successive lattice planes:

\[ 2d \cos(\theta) = m \lambda \]
\[ \Rightarrow \frac{2\pi}{\lambda} \left( m \frac{2\pi}{d} \right) \cos(\theta) = \frac{1}{2} \left( m \frac{2\pi}{d} \right)^2 \]
\[ \Rightarrow \mathbf{k} \cdot \mathbf{G} = \frac{|G|^2}{2} \]

Bragg Planes

Corresponding to every reciprocal lattice vector there is a Bragg plane in k-space that is a perpendicular bisector of that reciprocal lattice vector

Let's draw few of the Bragg planes for the square 2D reciprocal lattice corresponding to the reciprocal lattice vectors of the smallest magnitude
Bragg Planes and Higher Order Brillouin Zones

Bragg planes are shown for the square 2D reciprocal lattice corresponding to the reciprocal lattice vectors of the smallest magnitude.

Higher Order Brillouin Zones

The nth BZ can be defined as the region in k-space that is not in the (n-1)th BZ and can be reached from the origin by crossing at the minimum (n-1) Bragg planes.

The length (1D), area (2D), volume (3D) of BZ of any order is the same.

Appendix: Proof of the General Lattice FT Relation in 3D

This appendix gives proof of the FT relation:

\[ f(\hat{r}) = \sum_j \delta^3(\hat{r} - \hat{R}_j) \quad \leftrightarrow \quad f(\hat{k}) = \frac{(2\pi)^3}{\Omega_3} \sum_j \delta^3(\hat{k} - \hat{G}_j) \]

for the general case when the direct lattice primitive vectors are not orthogonal.

Let: \( \hat{R} = n_1 \hat{a}_1 + n_2 \hat{a}_2 + n_3 \hat{a}_3 \)

Define the reciprocal lattice primitive vectors as:

\[ \hat{b}_1 = 2\pi \frac{\hat{a}_2 \times \hat{a}_3}{\Omega_3} \quad \hat{b}_2 = 2\pi \frac{\hat{a}_3 \times \hat{a}_1}{\Omega_3} \quad \hat{b}_3 = 2\pi \frac{\hat{a}_1 \times \hat{a}_2}{\Omega_3} \]

Note: \( \hat{a}_j \cdot \hat{b}_k = 2\pi \delta_{jk} \)

Now we take FT:

\[ f(\hat{k}) = \int d^3\hat{r} \quad f(\hat{r}) \quad e^{-i\hat{k} \cdot \hat{r}} = \int d^3\hat{r} \quad \sum_j \delta^3(\hat{r} - \hat{R}_j) \quad e^{-i\hat{k} \cdot \hat{r}} = \sum_j \quad e^{-i\hat{k} \cdot \hat{R}_j} \]
Appendix: Proof

One can expand $\mathbf{k}$ in any suitable basis. Instead of choosing the usual basis:

$$\mathbf{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

I choose the basis defined by the reciprocal lattice primitive vectors:

$$\mathbf{k} = k_1 \mathbf{b}_1 + k_2 \mathbf{b}_2 + k_3 \mathbf{b}_3$$

Given that: $\bar{a}_j \cdot \mathbf{b}_k = 2\pi \delta_{jk}$

I get:

$$f(k) = \sum_j e^{-i \mathbf{k} \cdot \mathbf{R}_j} = \sum_{n_1 n_2 n_3} e^{-i \mathbf{k} \cdot (n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 + n_3 \mathbf{a}_3)}$$

$$= \sum_{m_1 m_2 m_3} \delta(k_1 - m_1) \delta(k_2 - m_2) \delta(k_3 - m_3)$$

Now:

$$\delta(k_1 - m_1) \delta(k_2 - m_2) \delta(k_3 - m_3) \propto \delta^3(\mathbf{k} - \mathbf{G})$$

where: $\mathbf{G} = m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2 + m_3 \mathbf{b}_3$

But we don’t know the exact weight of the delta function $\delta^3(\mathbf{k} - \mathbf{G})$

Appendix: Proof

Since: $\mathbf{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$ and $\mathbf{k} = k_1 \mathbf{b}_1 + k_2 \mathbf{b}_2 + k_3 \mathbf{b}_3$

This implies:

$$\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} b_{1x} & b_{2x} & b_{3x} \\ b_{1y} & b_{2y} & b_{3y} \\ b_{1z} & b_{2z} & b_{3z} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \tag{1}$$

Any integral over k-space in the form:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_1 \, dk_2 \, dk_3$$

can be converted into an integral in the form:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x \, dk_y \, dk_z$$

by the Jacobian of the transformation:

$$\begin{bmatrix} dk_x \\ dk_y \\ dk_z \end{bmatrix} \rightarrow \begin{bmatrix} \delta(k_x, k_y, k_z) \\ \delta(k_1, k_2, k_3) \end{bmatrix} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_1 \, dk_2 \, dk_3$$

Therefore:

$$\delta(k_1 - m_1) \delta(k_2 - m_2) \delta(k_3 - m_3) \propto \delta^3(\mathbf{k} - \mathbf{G})$$
Appendix: Proof

From (1) on previous slide:

\[
\frac{\partial (k_x, k_y, k_z)}{\partial (k_1, k_2, k_3)} = |\vec{b}_1 \cdot (\vec{b}_2 \times \vec{b}_3)| = \Pi_3 = \frac{(2\pi)^3}{\Omega_3}
\]

Therefore:

\[
f(k) = \sum_j e^{-i \vec{k} \cdot \vec{r}_j} = \sum_{m_1 m_2 m_3} \delta(k_1 - m_1) \delta(k_2 - m_2) \delta(k_3 - m_3)
\]

\[
= \frac{(2\pi)^3}{\Omega_3} \sum_j \delta^3(k - \bar{G}_j)
\]