PCSDE Models for Bivariate Power-Law Behavior

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Overview

1 Past work

2 1D PCSDE Model

3 2D PCSDE Models
   - 2D PCSDE Model with a Shared Poisson Counter
   - Models with Markov on-off Modulation
   - Model with Coupled Differential Equations

4 2D PCSDE Model with Brownian Motion (Ongoing work)

5 Conclusions
Degree distribution is not enough - the false Achilles heel of the Internet

Can one hear the shape of a complex network - spectral analysis in terms of the heat content function

Computational methods for very large matrices in eigenvalue range intervals
Degree distribution is not enough

Shan Lu et al. (UMass and Cornell)
Graphs with the same power law degree distribution

- Models for graphs with the same degree distribution
  - BA Model
  - Molly-Reed Model
  - Kalisky Model
  - Models of Grisi-Filho et al
    - Model A
    - Model B

Figure is from “Scale-Free Networks with the Same Degree Distributions: Different Structural Properties”, (J. H. H. Grisi-Filho, et al. 2013)
Alpha spectrum of power law graphs

Black: B-A model; Blue: M-R; Red: Kalisky; Green: Model A; Magenta: Model B
Upper Tail Power-Law Generator

\[ dX_t = \beta X_t dt + (x_0 - X_{t^-}) dN_t \]

- \( N_t \) is a Poisson counter with rate \( \lambda \).
- Stationary density:

\[ f_X(x) = \frac{\lambda}{\beta x_0} \left( \frac{x}{x_0} \right)^{-\frac{\lambda}{\beta} - 1}, \quad x \geq x_0 \]

- Complementary Cumulative Distribution Function (CCDF):

\[ \bar{F}_X(x) = \left( \frac{x}{x_0} \right)^{-\frac{\lambda}{\beta}}, \quad x \geq x_0. \]
SDE Model with both Poisson Counter and Browian Motion

\[ dX_t = \beta X_t dt + \sigma X_t dW_t + (x_0 - X_{t-}) dN_t \]

- Geometric Browian Motion (GBM) with Poisson Resetting.
- Stationary density: double-Pareto distribution [Reed, 2001].
- Power-law behavior in both tails.

Figure: Twitter out-degree distribution
2D Power-Law in Real Data [KONECT, 2013]:
- Social Networks: Youtube, Flickr, Livejournal, etc.

2D PCSDE model as an explanation of correlated power law behavior in social networks?
Model formulation

\[ dX_i = X_i dt + (1 - X_i)(dN_0 + dN_i), \quad i = 1, 2 \]

- \(N_0, N_1, \) and \(N_2\) are independent Poisson counters with rates \(\lambda_0, \lambda_1\) and \(\lambda_2\). Let \(\lambda_+ = \lambda_0 + \lambda_1 + \lambda_2\),

\[
f_{X_i}(x_i) = (\lambda_0 + \lambda_i)x_i^{-(\lambda_0+\lambda_i+1)}, \quad x_i \geq 1,
\]

\[
f_{X_1, X_2}(x_1, x_2) = \lambda_0 x_1^{-(\lambda_++1)} \delta(x_1 - x_2) + \lambda_1 x_1^{-(\lambda_++1)} f_{X_2}(x_2 x_1^{-1}) x_1^{-1}
\]
\[
+ \lambda_2 x_2^{-(\lambda_++1)} f_{X_1}(x_1 x_2^{-1}) x_2^{-1}, \quad x_1, x_2 \geq 1.
\]

- Tail behavior: \(P(X_2 > x | X_1 > x) = \frac{\bar{F}_{X_1, X_2}(x, x)}{\bar{F}_{X_1}(x)} = x^{1-\lambda_2} \xrightarrow{x \to \infty} 0.\)
Model formulation

\[ dX_i = X_i dt + (1 - X_i) ((1 - Y) dN_0 + Y dN_i), \quad i = 1, 2 \]

- Markov on-off Process \( Y_t \),
  \[ dY_t = (1 - Y_t) dM_1 - Y_t dM_2, \quad Y_0 \in \{0, 1\} \]

\( M_1 \) and \( M_2 \) are independent Poisson counters with rates \( \mu_1 \) and \( \mu_2 \).

- The shared Poisson counter \( N_0 \) is effective when \( Y_t = 0 \);
- The independent Poisson counters \( N_1 \) and \( N_2 \) are effective when \( Y_t = 1 \).
Use the characteristic function as in [JBGT, 2012];

Marginal and Joint CCDF:

\[
\bar{F}_{X_i}(x) = ax^{-A_i} b, \quad \bar{F}_{X_1, X_2}(x, x) = ax^{-A} b,
\]

where

\[
A_i = \begin{pmatrix} \lambda_0 & \lambda_i - \lambda_0 \\ -\mu_1 & \lambda_i + \mu_1 + \mu_2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda_0 & \sum_{i=1,2} \lambda_i - \lambda_0 \\ -\mu_1 & \sum_{i=1,2} (\lambda_i + \mu_i) \end{pmatrix},
\]

with \( a = (1, 0), \ b = (1, m(\infty))^T \) and \( m(\infty) = E[Y_\infty] = \frac{\mu_1}{\mu_1 + \mu_2} \).
Tail behavior: let $\xi^i_i$: eigenvalues of $A_i$; $\xi_\pm$: eigenvalues of $A$,

$$
\begin{align*}
\xi^{(i)}_\pm &= \frac{\lambda_0 + \lambda_i + \mu_1 + \mu_2}{2} \pm \frac{\sqrt{(\lambda_i - \lambda_0 + \mu_2 - \mu_1)^2 + 4\mu_1\mu_2}}{2} \\
\xi_\pm &= \frac{\lambda_+ + \mu_1 + \mu_2}{2} \pm \frac{\sqrt{(\lambda_1 + \lambda_2 - \lambda_0 + \mu_2 - \mu_1)^2 + 4\mu_1\mu_2}}{2}.
\end{align*}
$$

- Easy to check $\xi_- - \xi^{(1)}_- > 0$;
- $P(X_2 > x \mid X_1 > x) \sim Cx^{-(\xi_- - \xi^{(1)}_-)} \xrightarrow{x \to \infty} 0$;
- Still asymptotically independent.
Markov on-off Modulation II

Model formulation

\[ dy = (1 - y) dm_1 - y dm_2, \]
\[ dx_i = x_i dt + (1 - x_i)((1 - y)(dn_0 + dm_1) + y(dn_i + dm_2)). \]

- \( x_i \) resets when Markov on-off process \( y \) changes its state.

Marginal and Joint CCDF:

\[ F_{x_i}(x) = ax^{-A_i} b \quad F_{x_1,x_2}(x,x) = ax^{-A} b, \]

\[ A_i = \begin{pmatrix} \lambda_0 + \mu_1 & \lambda_i + \mu_2 - \lambda_0 - \mu_1 \\ 0 & \lambda_i + \mu_2 \end{pmatrix}, \]

\[ A = \begin{pmatrix} \lambda_0 + \mu_1 & \sum_{i=1,2} \lambda_i + \mu_2 - \lambda_0 - \mu_1 \\ 0 & \sum_{i=1,2} \lambda_i + \mu_2 \end{pmatrix}. \]
The spectral decomposition of $A$ and $A_i$ leads to

$$\bar{F}_{X_i}(x) = x^{-(\lambda_i + \mu_2)} m(\infty) + x^{-(\lambda_0 + \mu_1)} (1 - m(\infty)),$$

$$\bar{F}_{X_1, X_2}(x, x) = x^{-(\lambda_1 + \lambda_2 + \mu_2)} m(\infty) + x^{-(\lambda_0 + \mu_1)} (1 - m(\infty)).$$

Tail behavior: let $\lambda_1 = \lambda_2 = \lambda$, $\lambda_0 + \mu_1 = \lambda'_0$, $\lambda + \mu_2 = \lambda'$,

$$P(X_2 > x | X_1 > x) \xrightarrow{x \to \infty} \begin{cases} 1 & \lambda' > \lambda'_0 \\ \frac{\mu_2}{\mu_1 + \mu_2} & \lambda' = \lambda'_0 \\ 0 & \lambda' < \lambda'_0. \end{cases}$$

Tail dependence coefficient goes to 1 or 0;
The case when tail dependence coefficient is fractional is not robust.
Mixture of two models:

\[
\begin{cases}
    dX_i = X_i dt + (1 - X_i) dN_0, & \text{w.p. } 1 - m(\infty); \\
    dX_i = X_i dt + (1 - X_i) dN_i, & \text{w.p. } m(\infty), \\
\end{cases}
\]

\( N_0 \) with rate \( \lambda'_0 \) and \( N_i, \ i = 1, 2 \) with rate \( \lambda_1 = \lambda_2 = \lambda' \).

Tail behavior is determined by which model the observed large value more likely belongs to.

A model with stable fractional tail dependence?
Model formulation

\[
d \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 1 - X_1 \\ 0 \end{pmatrix} dN_1 + \begin{pmatrix} 0 \\ 1 - X_2 \end{pmatrix} dN_2
\]

Marginal tail: let \( X_n \) be the value of \( X_1(t) \) at the \( n^{th} \) arrival of the Poisson process \( N_2 \). Prove \( (X_n) \) satisfy a stochastic recursion

\[
X_{n+1} = A_{n+1} X_n + B_{n+1}, \quad n = 1, 2, \ldots
\]

Then, for a stationary random variable \( X \) satisfy

\[
X \overset{d}{=} AX + B,
\]

we have \( P(X > x) \sim Cx^{-\alpha}, \ x \to \infty. \ \alpha > 0 \) is such that \( EA^\alpha = 1 \).
Write the matrix

$$\beta = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = \lambda.$$ 

Note that the differential equation

$$dX(t) = \beta X(t) dt$$

has the solution

$$X(t) = e^{t\beta} X(0)$$

$$= \frac{1}{2} e^{t(1+\beta)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X(0)$$

$$+ \frac{1}{2} e^{t(1-\beta)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} X(0).$$

(1)
With (1), we compute $A$ as follows.

Let i.i.d. $(T_j) \sim \text{exp}(2\lambda)$ independent of $N \sim \text{Ge}(1/2)$.

$$A = \begin{cases} \frac{e^{T_1(1+\beta)}+e^{T_1(1-\beta)}}{2} & N = 0 \\ \frac{e^{T_1(1+\beta)}-e^{T_1(1-\beta)}}{2} \cdot \frac{e^{T_2(1+\beta)}-e^{T_2(1-\beta)}}{2} & N \geq 0 \\ \prod_{j=3}^{N+1} \frac{e^{T_j(1+\beta)}+e^{T_j(1-\beta)}}{2} & N \geq 0 \end{cases}$$

(2)

Solve $\alpha$,

$$\mathbb{E}A^\alpha = \frac{1}{2} I_1 + I_2 \frac{1}{4 - 2I_1} = 1,$$

where

$$I_1 = \frac{\lambda 2^{-\alpha}}{\beta} \int_0^1 z \frac{2\lambda - \alpha(1+\beta)}{2\beta}^{-1}(1 + z)^{\alpha} dz,$$

$$I_2 = \frac{\lambda 2^{-\alpha}}{\beta} B \left( \frac{2\lambda - \alpha(1+\beta)}{2\beta}, \alpha + 1 \right).$$
When $\beta = 0$, $\alpha = \lambda$.

- $\alpha$ decreases with $\beta$ increasing.

- $\alpha > 0$ exists when $E[\log A] < 0$.

**Figure:** $\alpha$ as a function of $\beta$
Let $X$ be a random variable with the stationary distribution of the value of $X_1(t)$ at the moment when the counter $N_2$ has an arrival.

Consider the combined counter $N_1 \cup N_2$. Its points are $W_1, W_2, \ldots$, with $(W_{n+1} - W_n)$ i.i.d, exp$(2\lambda)$.

The state of the system at these points has the stationary distribution

\[
\begin{pmatrix}
1 \\
X \\
1
\end{pmatrix} \text{ w.p. } \frac{1}{2}
\]

\[
\begin{pmatrix}
X \\
1
\end{pmatrix} \text{ w.p. } \frac{1}{2}
\]

Solution in (1) and (3) give the stationary distribution.
Joint tail: let $T \sim \exp(2\lambda)$ and given $T = t$, $u \sim \mathbb{U}(0, t)$,

$$V = \frac{e^{u(1+\beta)} - e^{u(1-\beta)}}{2}; \quad W = \frac{e^{u(1+\beta)} + e^{u(1-\beta)}}{2}.$$  

In the stationary regime,

$$(X_1, X_2) \overset{d}{=} \begin{cases} (XV + W, XW + V) & \text{w.p. } \frac{1}{2} \\ (XW + V, XV + W) & \text{w.p. } \frac{1}{2} \end{cases}.$$  \hfill (4)

where $P(X > x) \sim Cx^{-\alpha}$.

Tail behavior: with Breiman’s lemma [Breiman, 1965],

$$\lim_{x \to \infty} P(X_2 > x \mid X_1 > x) = \frac{2E[V^\alpha]}{E[V^\alpha] + E[W^\alpha]},$$
Numerical results for Tail Dependence

- When $\beta = 0$, tail dependence coefficient equals 0.
- Tail dependent coefficient increases as $\beta$ increases.
- Tail dependent coefficient approaches 1 when $\alpha$ approaches 0.

**Figure:** Tail dependence coefficient as a function of $\beta$
Model formulation

\[
d \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & \beta_1 \\ \beta_2 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 1 - X_1 \\ 0 \end{pmatrix} dN_1 + \begin{pmatrix} 0 \\ 1 - X_2 \end{pmatrix} dN_2
\]

- \( A \) is the same as in (2) with \( \beta = \sqrt{\beta_1 \beta_2} \).
- Let \( V_1 = \sqrt{\frac{\beta_1}{\beta_2}} V \) and \( V_2 = \sqrt{\frac{\beta_2}{\beta_1}} V \),

\[
\lim_{x \to \infty} \frac{P(X_1 > x, X_2 > x)}{P(X_i > x)} = \frac{E \left[ \min (W, V_1)^\alpha \right] + E \left[ \min (W, V_2)^\alpha \right]}{E[V_i^\alpha] + E[W^\alpha]}.
\]
Numerical Results

Let $\lambda = 1/4$, fix $\beta_1 = 0.001$, the marginal tail $\alpha$ decreases with the increasing of $\beta_2$ value.

The tail dependence coefficients with $X_1$ given or with $X_2$ given are different when $\beta_1 \neq \beta_2$. 

**Figure:** $\alpha$ as a function of $\beta$

**Figure:** Tail dependence coefficient as a function of $\beta$
For Real Data in Social Networks?

- The model with coupled differential equations has a feature not observed in 2D power-law data we know (but could be useful in modeling the prey-predator power law?)
  \( \lambda = 2, \beta = 0.2, \alpha = 1.9203. \)

- Go back to a single Poisson counter (two rare events occur together in the most likely way - the same cause)

- PCSDE model describe the expected degree growth.

- A Brownian motion component may help describing the randomness of degree growth.

Figure: Real 2D data in Social Network
Model Formulation

\begin{align*}
    dX_1 &= \beta_1 X_1 \, dt + \sigma_1 X_1 \, dW_1 + (1 - X_1) \, dN_0 \\
    dX_2 &= \beta_2 X_2 \, dt + \sigma_2 X_2 \, dW_2 + (1 - X_2) \, dN_0
\end{align*}

- Based on 1D Geometric Brownian Motion with Poisson resetting.
- Let $Y_i = \log X_i$,

\[ dY_i = \left( \beta_i - \frac{1}{2} \sigma_i^2 \right) \, dt + \sigma_i \, dW_i - Y_i \, dN_0, \quad i = 1, 2. \]

- Given $t \sim \exp(\lambda_0)$,

\[ X_i(t) = \exp \left( \left( \beta_i - \frac{1}{2} \sigma_i^2 \right) t + \sigma_i W_i(t) \right), \quad i = 1, 2. \]
- Synthetic data generated by the model with Brownian Motion I with different $\sigma$ values.
- The samples from this model do not fit well to the real data in social networks.

**Figure:** Synthetic data generated by the model with Brownian Motion I
How to modify the model to fit real data in social networks?

- With preferential attachment, each node is selected to be the target node with probability proportional to its current degree $D$.

- Think of dividing the node into $D$ nodes with degree 1, each node will be selected as a target node with equal probability $p$.

- The new degree added to this node $d \sim B(D, p)$ with mean $Dp$ and variance $Dp(1-p)$.

- A reasonable approximation to $B(D, p)$ when $D$ is large is given by the normal distribution $\mathcal{N}(Dp, Dp(1-p))$.

- The variance is proportional to $D$. So, the standard deviation should be proportional to $\sqrt{D}$. 

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Model Formulation

\[ dX_1 = \beta_1 X_1 dt + \sigma_1 \sqrt{X_1} dW_1 + (1 - X_1) dN_0 \]
\[ dX_2 = \beta_2 X_2 dt + \sigma_2 \sqrt{X_2} dW_2 + (1 - X_2) dN_0 \]

- Let \( Y_i = \sqrt{X_i} \),

\[ dY_i = \left( \frac{1}{2} \beta_i Y_i - \frac{1}{8} \sigma_i^2 \frac{1}{Y_i} \right) dt + \frac{1}{2} \sigma_i dW_i + (1 - Y_i) dN_0, \ i = 1, 2. \]

- For the tail, \( Y_i \to \infty \),

\[ dY_i = \frac{1}{2} \beta_i Y_i dt + \frac{1}{2} \sigma_i dW_i + (1 - Y_i) dN_0, \ i = 1, 2. \]
- Synthetic data:

![Figure: Synthetic data generated by the model with Brownian motion II](image1)

- Comparing to real data:

![Figure: Comparing to real data](image2)
Other works

- David Mumford (1974 Fields medalist) in “Self-similarity of image statistics and image models”: The hypothesis that natural images of the world, treated as a single large database, have renormalization invariant statistics has received remarkable confirmation from many quite distinct tests.

- Understanding the origin and generative mechanisms for scaling law in natural images is very important in developing more intelligent image processing methods.

- Our hypothesis is that human and animals have to be able to extract the same features against resolution blurring for survival. Mathematical study of this could provide new techniques in addition to the Scale Invariant Feature Transform (SIFT).
Conclusions

- We present a modulated sharing Poisson counter model with tail dependence coefficient to be 0 or 1. By adding a Brownian motion component to this model, we generate samples distributed like the ones observed in social networks.

- We also propose a model with fractional tail dependence coefficient. This model is interesting theoretically; however, the distribution of the samples generated by this model do not fit to the real data we know.
References

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L. Breiman (1965)
On Some Limit Theorems Similar to the Arc-Sin Law
Suppose that $X$ and $Y$ are two independent nonnegative random variables such that $P(X > x)$ is regularly varying of index $-\alpha$, $\alpha \geq 0$, and that $E[Y^{\alpha + \epsilon}] < \infty$ for some $\epsilon > 0$. Then

$$P(XY > x) \sim E[Y^\alpha]P(X > x)$$
Thank You!