Fourier Transform - A Brief Summary

Adapted from notes by Michael Braun

This handout is intended as a refresher on the properties of Fourier Transforms. More detailed discussions can be found in

E. Hecht, *Optics*, 2^{nd} ed, Chapter 11
E. G. Stewart, *FOURIER OPTICS, An Introduction*

There are also some interesting web-based summaries of Fourier Transforms such as: *Kevin Cowtan's Book of Fourier* has graphical examples, and applications to crystallography: http://www.ysbl.york.ac.uk/~cowtan/fourier/fourier.html

Definition

Let $f(x)$ be a function of some independent variable x (this may be time or spatial position). A Fourier transform maps the function $f(x)$ into another function $F(s)$ defined in the Fourier domain (the independent variable s may stand for temporal or spatial frequency).

$$F(s) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{i2\pi s x}\, dx \qquad \text{-(1)}$$

In these notes we follow the usual convention, which uses uppercase letters to denote Fourier transforms, i.e. $F(s) = \mathcal{F}\{f(x)\}$, $G(s) = \mathcal{F}\{g(x)\}$ and so on. Physicists often use the variable $k=2\pi s$ instead. For crystallography, s is more convenient since Bragg's law then reduces to $d=1/s$.

Conditions for Existence

The Fourier transform $F(s)$ exists if

1. $f(x)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \qquad \text{-(2)}$$

2. any discontinuities in $f(x)$ are finite, and

3. $f(x)$ has only a finite number of discontinuities and only a finite number of maxima and minima in any finite interval

Inverse Transform

The inverse Fourier transform in defined as

$$f(x) = \mathcal{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} F(s)\, e^{-i2\pi s x}\, ds\,. \qquad \text{-(3)}$$

If $f(x)$ is discontinuous, it should be replaced in the definition above with the mean of the "from above" and "from below" limits

$$\frac{f(x+) + f(x-)}{2}.$$

Linearity

Given Fourier transforms $F(s) = \mathcal{F}\{f(x)\}$, $G(s) = \mathcal{F}\{g(x)\}$ and constants a and b,

$$\mathcal{F}\{a\, f(x) + b\, g(x)\} = a\, F(s) + b\, G(s)\,. \qquad \text{-(4)}$$

Similarity

Given $F(s) = \mathcal{F}\{f(x)\}$ and a constant a,

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|} F(\frac{s}{a})\,. \qquad \text{-(5)}$$

Shift

Given $F(s) = \mathcal{F}\{f(x)\}$ and a shift a in the x–domain,

$$\mathcal{F}\{f(x-a)\} = e^{i2\pi a s} F(s)\,. \qquad \text{-(6)}$$

Similarly, for a shift α in the Fourier domain,

$$\mathcal{F}^{-1}\{F(s-\alpha)\} = e^{-i2\pi \alpha s} f(x)\ . \qquad \text{-(7)}$$

Derivative

Let $F(s) = \mathcal{F}\{f(x)\}$ and denote $f' = \frac{df}{dx}$. Then

$$\mathcal{F}\{f'(x)\} = -i2\pi s F(s) \qquad \text{-(8)}$$

and for the *n*th derivative,

$$\mathcal{F}\{f^{(n)}(x)\} = (-i2\pi s)^n F(s)\,. \qquad \text{-(9)}$$

Convolution

Given two functions $f(x)$ and $g(x)$, whose Fourier transforms are $F(s)$ and $G(s)$, respectively, the Fourier transform of the convolution of the two functions is the product of their Fourier transforms

$$\mathcal{F}\{f(x) * g(x)\} = F(s)\, G(s)\,, \qquad \text{-(10)}$$

where the convolution is defined as:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi \qquad \text{-(11)}$$

Crosscorrelation

Define the crosscorrelation of the two functions $f(x)$ and $g(x)$ as follows:

$$f(x) \star g(x) = \int_{-\infty}^{\infty} f^*(\xi)\, g(x+\xi)\, d\xi \qquad \text{-(12)}$$

(Here, the * denotes complex conjugation, where i goes to $-i$.)

Then the crosscorrelation is related to the convolution by:

$$f(x) \star g(x) = f^*(-x) * g(x)\,, \qquad \text{-(13)}$$

It follows that, given $F(s) = \mathcal{F}\{f(x)\}$ and $G(s) = \mathcal{F}\{g(x)\}$,

$$\mathcal{F}\{f(x) \star g(x)\} = F^*(s)\, G(s) \qquad \text{-(14)}$$

Autocorrelation

From above, it follows that the Fourier transform of the crosscorrelation of a function with itself (autocorrelation) is given by the squared modulus of its Fourier transform,

$$\mathcal{F}\{f(x) * f(x)\} = |F(s)|^2 \qquad \text{-(15)}$$

Raleigh's Theorem

The integral of the squared modulus of a function is equal to the integral of the squared modulus of its transform. Thus if $F(s)$ is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds \qquad \text{-(16)}$$

Power

Let $F(s)$ and $G(s)$ be the Fourier transforms of $f(x)$ and $g(x)$, respectively. Then

$$\int_{-\infty}^{\infty} f(x) g(x)^* \, dx = \int_{-\infty}^{\infty} F(s) G(s)^* \, ds \qquad \text{-(17)}$$

Area

The area under a function $f(x)$ is given by

$$\int_{-\infty}^{\infty} f(x) \, dx = F(0) \qquad \text{-(18)}$$

Moments

The first moment of a function $f(x)$ is given by

$$\int_{-\infty}^{\infty} x \, f(x) \, dx = \frac{-i}{2\pi} \frac{dF(0)}{ds} \qquad \text{-(19)}$$

For the nth moment, we have

$$\int_{-\infty}^{\infty} x^n \, f(x) \, dx = \left(\frac{-i}{2\pi} \right)^n \frac{d^n F(0)}{ds^n} \qquad \text{-(20)}$$

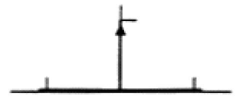
Some useful functions

The table below defines and illustrates several functions which appear often in Fourier transforms. The tick marks indicate points $(0, \pm 1)$ and $(\pm 1, 0)$.

Dirac delta

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

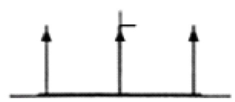
and $\int_{-\infty}^{\infty} \delta(x) dx = 1$



Comb

$$\text{comb}(x) = \sum_{-\infty}^{\infty} \delta(x - n)$$

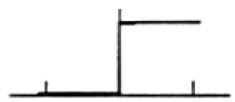
also denoted $\text{III}(x)$



Step

$$u(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$$

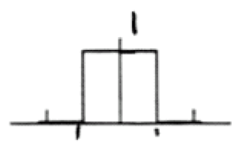
also known as Heaviside unit step function $H(x)$



Rectangle

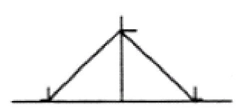
$$\text{rect}(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$$

also denoted $\Pi(x)$



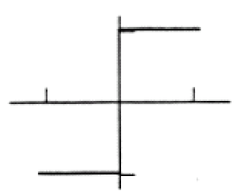
Triangle

$$\Lambda(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$



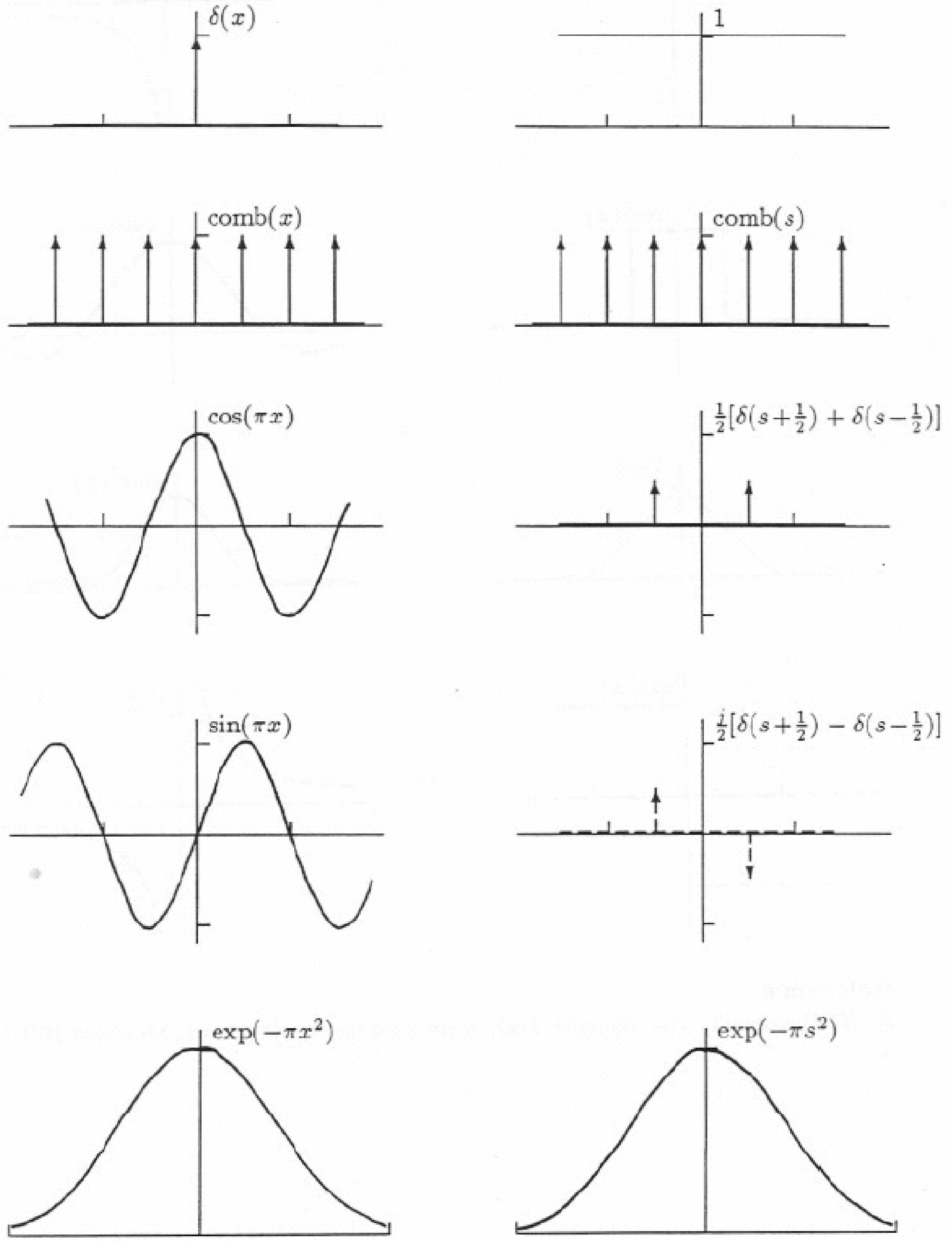
Sign

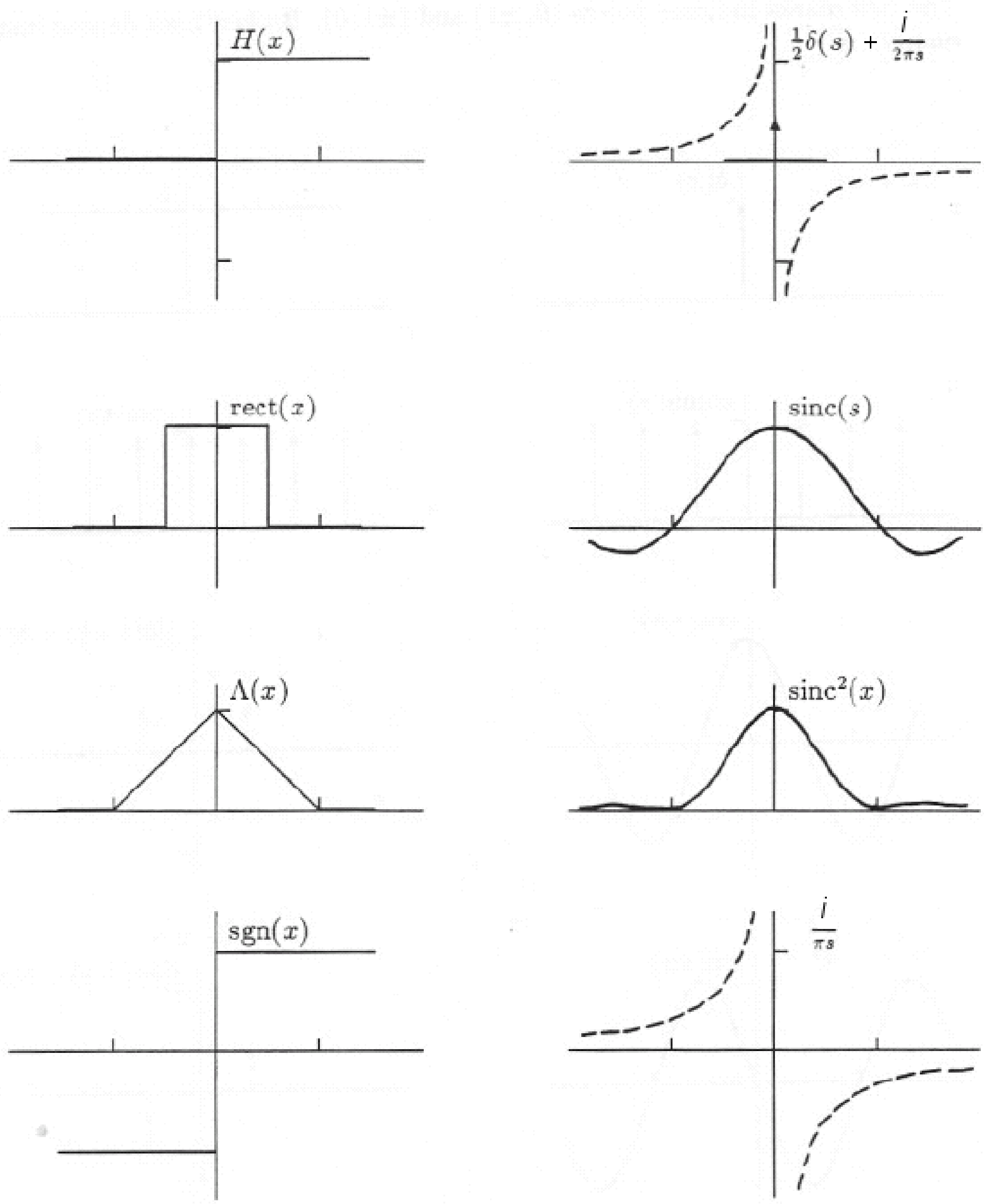
$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$$



Some Fourier Transform Pairs

The table below illustrates some commonly encountered Fourier transform pairs. The tick marks indicate points $(0, \pm 1)$ and $(\pm 1, 0)$. Broken lines denote imaginary quantities.





Reference

R. N. Bracewell, *The Fourier Transform and its Applications,* McGraw-Hill 1978.