2D Fourier Transform

Definition

$F(u,v) = \mathcal{F}\{f(x,y)\} = \int\int_{-\infty}^{\infty} f(x,y)\, e^{i2\pi(ux+vy)}\, dxdy$

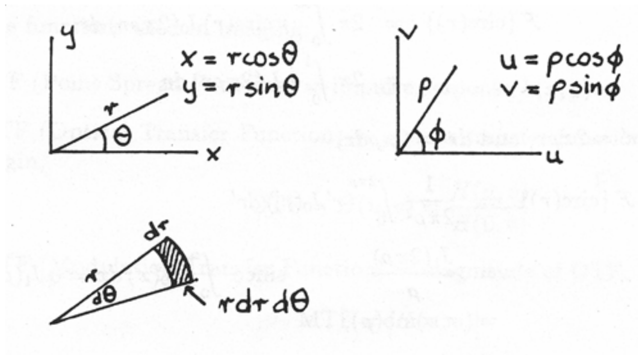
where x, y are rectangular spatial coordinates and u, v are spatial frequencies.

We can rewrite this as two 1D transforms:

$F(u,v) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, e^{i2\pi ux}\, dx \right] e^{i2\pi vy}\, dy$

i.e. for each y take a 1D FT with respect to x, then take a 1D FT with respect to y.

Polar Coordinates



Since

$$u x + v y = \rho r(\cos\phi\cos\theta + \sin\phi\sin\theta)$$
$$= \rho r\cos(\phi - \theta),$$

then the Fourier transform in polar coordinates is

$$F(\rho,\phi) = \mathcal{F}\{f(r,\theta)\} = \int_0^{2\pi}\int_0^{\infty} f(r,\theta) e^{i2\pi\rho r\cos(\phi-\theta)} r\,dr\,d\theta$$

and the inverse transform is

$$f(r,\theta) = \mathcal{F}^{-1}\{F(\rho,\phi)\} = \int_0^{2\pi}\int_0^{\infty} F(\rho,\phi) e^{-i2\pi\rho r\cos(\phi-\theta)} \rho\,d\rho\,d\phi$$

Separability in Polar Coordinates

A function $f(r,\theta)$ is separable in polar coordinates if it can be written in the form

$$f(r,\theta) = f_r(r)\, f_\theta(\theta).$$

Suppose such an f is circularly symmetric, with $f_\theta(\theta) = 1$. As a specific example, we will consider a 2D circular (or cylinder) function:

$$circ(r) = \begin{cases} 1 & r < 1 \\ 1/2 & r = 1 \\ 0 & r > 1 \end{cases}$$

The Fourier transform of a circularly symmetric function is

$$F(\rho,\phi) = 2\pi\int_0^\infty r \; f_r(r) J_0(2\pi\rho r) dr \, .$$

This is also known as the Hankel transform of order zero and as the Fourier-Bessel transform. The function J_0 is the zero order Bessel function of the first kind defined as

$$J_0(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{ia\cos(\theta-\phi)} d\theta \, .$$

It oscillates like a damped cosine.

Continuing with our specific example, the Fourier transform of $circ(r)$ is

$$\mathcal{F}\{circ(r)\} = 2\pi\int_0^\infty r \; circ(r) \, J_0(2\pi\rho r) \, dr = 2\pi\int_0^1 r \, J_0(2\pi\rho r) \, dr \, .$$

Substitute $r' = 2\pi\rho r$, and $dr' = 2\pi\rho dr$ to find:

$$\mathcal{F}\{circ(r)\} = \frac{1}{2\pi\rho^2}\int_0^{2\pi\rho} r' \; J_0(r') \, dr' = \frac{J_1(2\pi\,\rho)}{\rho} \;\; , \qquad \text{since } \int_0^\alpha x \; J_0(x) \, dx = \alpha \, J_1(\alpha)$$

where J_1 is the first order Bessel function of the first kind, similar to a damped sinusoid.

The function $somb(\rho)$ or $sombrero$ (also known as Mexican hat, Bessinc, and jinc) is defined as

$$somb(\rho) = \frac{2J_1(2\pi\rho)}{2\pi\rho} \, ,$$

and is pictured on the right side below. Thus, the Fourier transform of $circ(r)$ is proportional to a sombrero function of ρ, the radial coordinate in frequency space:

$$\mathcal{F}\{circ(r)\} = \pi \, somb(\rho) \, .$$