

Mathematics of Compressive Sensing: Sparse recovery using ℓ_1 minimization

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Sparse recovery and restricted isometries

We say that a $M \times N$ matrix Φ is a restricted isometry for sets of size $2S$ if there exists a $0 \leq \delta_{2S} < 1$ such that

$$(1 - \delta_{2S})\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_{2S})\|x\|_2^2 \quad \text{for all } 2S\text{-sparse } x$$

It is clear that if Φ is $2S$ -RIP and $y = \Phi x_0$, where x_0 is an S -sparse vector, then we can recovery x_0 by solving

$$\min_x \|x\|_0 \quad \text{subject to} \quad \Phi x = y.$$

Now we will see that this condition also means we can recovery by minimizing the ℓ_1 norm...

Our goal

Theorem: Let Φ be an $M \times N$ matrix that is an approximate isometry for $3S$ -sparse vectors. Let x_0 be an S -sparse vector, and suppose we observe $y = \Phi x_0$. Given y , the solution to

$$\min_x \|x\|_1 \quad \text{subject to} \quad \Phi x = y$$

is *exactly* x_0 .

Moving to the solution

$$\min_x \|x\|_1 \quad \text{such that} \quad \Phi x = y$$

Call the solution to this x^\sharp . Set

$$h = x^\sharp - x_0.$$

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$$h = x^\sharp - x_0.$$

Two things must be true:

- $\Phi h = 0$
Simply because both x^\sharp and x_0 are feasible: $\Phi x^\sharp = y = \Phi x_0$
- $\|x_0 + h\|_1 \leq \|x_0\|_1$
Simply because $x_0 + h = x^\sharp$, and $\|x^\sharp\|_1 \leq \|x_0\|_1$

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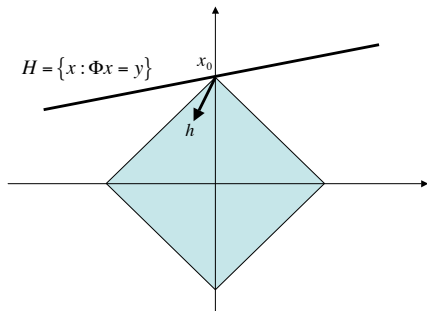
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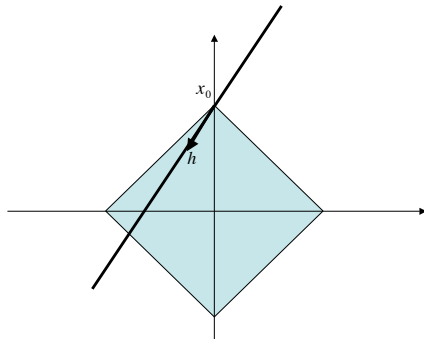
We'll show that if Φ is 3S-RIP, then these conditions are *incompatible* unless $h = 0$

Geometry

SUCCESS



FAILURE



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- $\Phi h = 0$
- $\|x_0 + h\|_1 \leq \|x_0\|_1$

Cone condition

For $\Gamma \subset \{1, \dots, N\}$, define $h_\Gamma \in \mathbb{R}^N$ as

$$h_\Gamma(\gamma) = \begin{cases} h(\gamma) & \gamma \in \Gamma \\ 0 & \gamma \notin \Gamma \end{cases}$$

Let Γ_0 be the support of x_0 . For any “descent vector” h , we have

$$\|h_{\Gamma_0^c}\|_1 \leq \|h_{\Gamma_0}\|_1$$

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Why? The triangle inequality..

$$\begin{aligned} \|x_0\|_1 &\geq \|x_0 + h\|_1 = \|x_0 + h_{\Gamma_0} + h_{\Gamma_0^c}\|_1 \\ &\geq \|x_0 + h_{\Gamma_0^c}\|_1 - \|h_{\Gamma_0}\|_1 \\ &= \|x_0\|_1 + \|h_{\Gamma_0^c}\|_1 - \|h_{\Gamma_0}\|_1 \end{aligned}$$

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$$\|h_{\Gamma_0^c}\|_1 \leq \|h_{\Gamma_0}\|_1$$

We will show that if Φ is 3S-RIP, then

$$\Phi h = 0 \quad \Rightarrow \quad \|h_{\Gamma_0}\|_1 \leq \rho \|h_{\Gamma_0^c}\|_1$$

for some $\rho < 1$, and so $h = 0$.

Some basic facts about ℓ_p norms

- $\|h_\Gamma\|_\infty \leq \|h_\Gamma\|_2 \leq \|h_\Gamma\|_1$
- $\|h_\Gamma\|_1 \leq \sqrt{|\Gamma|} \cdot \|h_\Gamma\|_2$
- $\|h_\Gamma\|_2 \leq \sqrt{|\Gamma|} \cdot \|h_\Gamma\|_\infty$

Dividing up $h_{\Gamma_0^c}$

Recall that Γ_0 is the support of x_0

Fix $h \in \text{Null}(\Phi)$. Let

$\Gamma_1 =$ locations of $2S$ largest terms in $h_{\Gamma_0^c}$,

$\Gamma_2 =$ locations next $2S$ largest terms in $h_{\Gamma_0^c}$,

\vdots

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Then

$$0 = \|\Phi h\|_2 = \left\| \Phi \left(\sum_{j \geq 0} h_{\Gamma_j} \right) \right\|_2 \geq \|\Phi(h_{\Gamma_0} + h_{\Gamma_1})\|_2 - \left\| \sum_{j \geq 2} \Phi h_{\Gamma_j} \right\|_2$$

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Applying the $3S$ -RIP gives

$$\begin{aligned} \sqrt{1 - \delta_{3S}} \|h_{\Gamma_0} + h_{\Gamma_1}\|_2 &\leq \|\Phi(h_{\Gamma_0} + h_{\Gamma_1})\|_2 \\ &\leq \sum_{j \geq 2} \|\Phi h_{\Gamma_j}\|_2 \leq \sum_{j \geq 2} \sqrt{1 + \delta_{2S}} \|h_{\Gamma_j}\|_2 \end{aligned}$$

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Applying the $3S$ -RIP gives

$$\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \leq \sqrt{\frac{1 + \delta_{2S}}{1 - \delta_{3S}}} \sum_{j \geq 2} \|h_{\Gamma_j}\|_2$$

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$$\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \leq \sqrt{\frac{1 + \delta_{2S}}{1 - \delta_{3S}}} \sum_{j \geq 2} \sqrt{2S} \|h_{\Gamma_j}\|_\infty$$

since $\|h_{\Gamma_j}\|_2 \leq \sqrt{2S} \|h_{\Gamma_j}\|_\infty$

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Then

$$\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \leq \sqrt{\frac{1 + \delta_{2S}}{1 - \delta_{3S}}} \sum_{j \geq 1} \frac{1}{\sqrt{2S}} \|h_{\Gamma_j}\|_1$$

since $\|h_{\Gamma_j}\|_\infty \leq \frac{1}{2S} \|h_{\Gamma_{j-1}}\|_1$

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Which means

$$\|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \leq \sqrt{\frac{1 + \delta_{2S}}{1 - \delta_{3S}}} \frac{\|h_{\Gamma_0^c}\|_1}{\sqrt{2S}}$$

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Working to the left

$$\|h_{\Gamma_0}\|_2 \leq \|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \leq \sqrt{\frac{1 + \delta_{2S}}{1 - \delta_{3S}}} \frac{\|h_{\Gamma_0^c}\|_1}{\sqrt{2S}}$$

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Working to the left

$$\frac{\|h_{\Gamma_0}\|_1}{\sqrt{S}} \leq \|h_{\Gamma_0}\|_2 \leq \|h_{\Gamma_0} + h_{\Gamma_1}\|_2 \leq \sqrt{\frac{1 + \delta_{2S}}{1 - \delta_{3S}}} \frac{\|h_{\Gamma_0^c}\|_1}{\sqrt{2S}}$$

Wrapping it up

We have shown

$$\begin{aligned}\|h_{\Gamma_0}\|_1 &\leq \sqrt{\frac{1 + \delta_{2S}}{1 - \delta_{3S}}} \sqrt{\frac{S}{2S}} \|h_{\Gamma_0^c}\|_1 \\ &= \rho \|h_{\Gamma_0^c}\|_1\end{aligned}$$

for

$$\rho = \sqrt{\frac{1 + \delta_{2S}}{2(1 - \delta_{3S})}}$$

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Taking $\delta_{2S} \leq \delta_{3S} < 1/3 \Rightarrow \rho < 1$.

SUCCESS!!

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