Homework# 1: Short course on sparse recovery and compressed sensing

October 2013

Reading:

The course notes, along with the homework assignments and example files, can be found at http://users.ece.gatech.edu/~justin/Tsinghua-Oct13/.

Freely available soft copies of the papers below can easily be tracked down online.

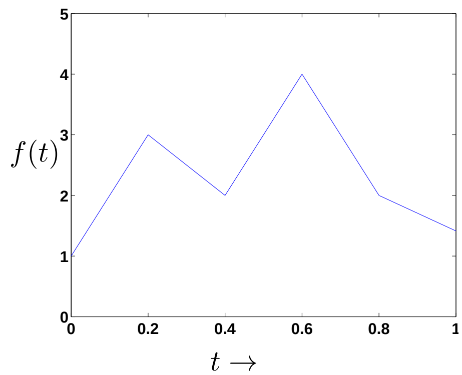
Lec. 1: For an excellent introduction to orthobasis expansions, see Young's *An Introduction to Hilbert Space*, http://amzn.to/X43teH

Lec. 2: For an extensive (and very mathematical) treatment of how sparsity is used in signal processing, see Mallat's *A Wavelet Tour of Signal Processing*, http://amzn.to/X44ovL

Lec. 3: The Mallat book is a good reference for this, too.

1. Write a one paragraph summary of what we talked about today. Use complete sentences, and avoid equations as much as possible.

2. Let PL_n be the space of functions on $[0,1]$ that are continuous and piecewise linear with knots at k/n, $k = 0, 1, \ldots, n$. An example of an $f \in \mathrm{PL}_5$ is shown below.



It should be clear that PL_n has dimension $n+1$, and that

$$\phi_0(t) = \begin{cases} \sqrt{3n}\,(1 - tn) & 0 \le t \le 1/n \\ 0 & \text{otherwise} \end{cases},$$

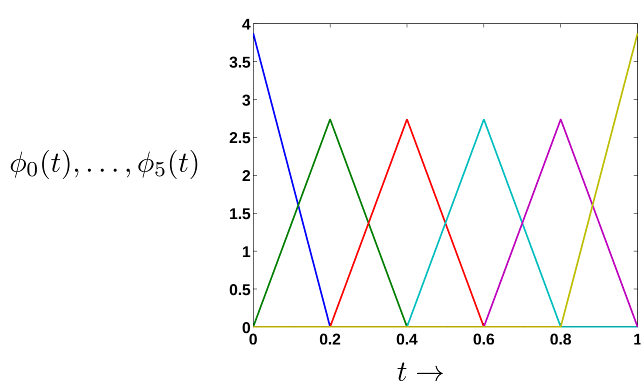
$$\phi_k(t) = \begin{cases} \sqrt{3n/2}\,(nt - k + 1) & (k-1)/n \le t \le k/n \\ \sqrt{3n/2}(k + 1 - nt) & k/n \le t \le (k+1)/n \\ 0 & \text{otherwise} \end{cases}, \quad k = 1, \ldots, n-1,$$

$$\phi_n(t) = \begin{cases} \sqrt{3n}(nt - n + 1) & (n-1)/n \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a basis for PL_n. For example, the signal in PL_5 above can be written as

$$f(t) = \frac{1}{\sqrt{15}} \cdot \phi_0(t) + \frac{3\sqrt{2}}{\sqrt{15}} \cdot \phi_1(t) + \frac{2\sqrt{2}}{\sqrt{15}}\phi_2(t) + \frac{4\sqrt{2}}{\sqrt{15}}\phi_3(t) + \frac{2\sqrt{2}}{\sqrt{15}}\phi_4(t) + \frac{1}{\sqrt{15}}\phi_5(t).$$

where the six basis functions are sketched on the same set of axes below:



In general we can write $f \in \mathrm{PL}_n$ as $f(t) = \sum_{k=0}^{n} \alpha_k \phi_k(t)$ where the α_k are just scaled samples of $f(t)$ taken at the knots $t = k/n$.

(a) Let $\Phi : \mathrm{PL}_n \to \mathbb{R}^{n+1}$ be the linear operator that maps such a continuous piecewise linear function to its (standard Euclidean) inner products against the ϕ_k:

$$(\Phi[f])_k = \langle f, \phi_k \rangle, \quad k = 0, \ldots, n.$$

The adjoint $\Phi^* : \mathbb{R}^{n+1} \to \mathrm{PL}_n$ synthesizes a signal from a set of coefficients $\alpha \in \mathbb{R}^{n+1}$ using

$$(\Phi^*[\alpha])(t) = \sum_{k=0}^{n} \alpha_k \phi_k(t).$$

Then the operator $\Phi\Phi^* : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ can be represented by an $n+1 \times n+1$ matrix. Write an expression for each entry $(\Phi\Phi^*)_{k,m}$, $0 \le k, m \le n$ in this matrix.

(b) Write a Matlab function that computes the matrix $\Phi\Phi^*$ for a fixed n. Use this function along with the `eig` command to calculate the frame bounds for this basis. That is, find the largest A and smallest B such that

$$A\|f\|_{L_2}^2 \;\le\; \|\Phi[f]\|_{\ell_2}^2 = \sum_{k=0}^{n} |\langle f, \phi_k \rangle|^2 \;\le\; B\|f\|_{L_2}^2.$$

for $n = 5, 10,$ and 100. (Recall that $\|\Phi\| = \|\Phi^*\|$, where $\|\cdot\|$ is the operator norm.) Report the three answers above, and turn in a printout of your code. Also turn in a plot the spectrum (eigenvalues) of $\Phi\Phi^*$ for $n = 100$.

(c) Write a Matlab script that calculates and plots the dual basis vectors $\{\tilde{\phi}_k\}$ for $n = 5$. The operator Φ is not a matrix, so you cannot just invert it. But you could easily have Matlab plot any linear combination of the ϕ_k. So you have to figure out how to find $\{\alpha_{k,j}$ so that $\tilde{\phi}_k(t) = \sum_j \alpha_{k,j}\phi_j(t)$. Turn in a print out of your script and the plots (on six separate axes, but all on one page, please).

3. Using MATLAB, implement the matching pursuit and orthogonal matching pursuit algorithms. Run it on the signal `f` and dictionary Ψ (which was chosen randomly) in `hw1problem3.mat`. Turn in your code and a stem plot of the current estimate f_k and a stem plot of the weights being used in the linear combination of columns for the current estimate f_k for the first 10 iterations. (The signal f_k has 32 entries here while the weighting vector should have 64 entires. You might need to think a little bit about how to generate the weights for OMP.)