Homework# 2: Short course on sparse recovery and compressed sensing

October 2013

Reading:

Lec. 4: A good supplement to this material is Donoho and Huo's "Uncertainty Principles and Ideal Atomic Decomposition," http://bit.ly/UfBfiG

Lec. 5: There is a great series of papers in the March 2008 issue of *IEEE Signal Processing Magazine* that overview the theory of compressed sensing from several different perspective, http://bit.ly/UfBz0X.

Problems:

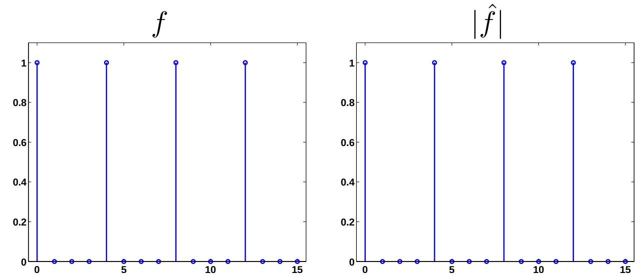
1. Write a one paragraph summary of what we talked about today. Use complete sentences, and avoid equations as much as possible.

2. Let $f \in \mathbb{C}^{16}$ and let $\hat{f}$ be the discrete Fourier transform of f:

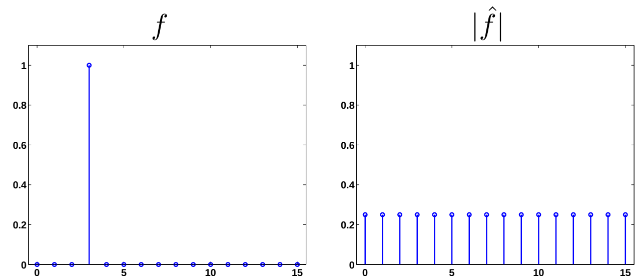
$$\hat{f}[\omega] = \sum_{k=0}^{15} f[k] e^{-j2\pi\omega k/n}, \quad \text{for } \omega = 0, 1, \ldots, 15.$$

Which of the following are *possible* time/frequency pairs for $f, \hat{f}$?

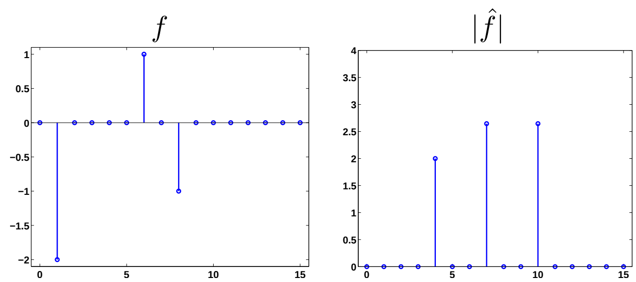
(a)



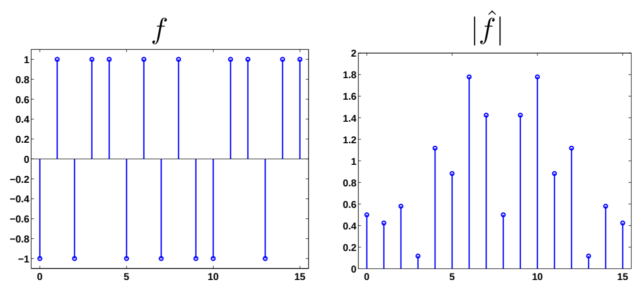
(b)



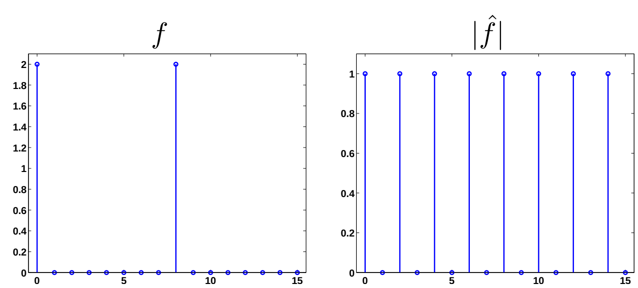
(c)



(d)



(e)



3. Consider the $n \times 2n$ dictionary Φ composed of two orthobases for $\mathbb{R}^n$, Ψ^1 and Ψ^2:

$$\Phi = \begin{bmatrix} \Psi^1 & \Psi^2 \end{bmatrix}.$$

Suppose that the maximum inner product between any column from Ψ^1 and any column from Ψ^2 is at most

$$\max_{\substack{\psi_1 \in \Psi^1 \\ \psi_2 \in \Psi^2}} |\langle \psi_1, \psi_2 \rangle| \ \leq \ 3/\sqrt{n}.$$

Let Γ_1 and Γ_2 be subsets of $\{1, 2, \ldots, n\}$, and consider the $n \times (|\Gamma_1| + |\Gamma_2|)$ submatrix

$$\Phi_\Gamma = \begin{bmatrix} \Psi^1_{\Gamma_1} & \Psi^2_{\Gamma_2} \end{bmatrix}.$$

(a) Find M such that

$$\Phi_\Gamma^* \Phi_\Gamma = I + \begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix}.$$

(b) What is the maximum possible value of an entry in M?

(c) Set

$$G = \begin{bmatrix} 0 & M \\ M^* & 0 \end{bmatrix}.$$

It is a fact that $\|G\| \leq \|M\|$ (where here $\|\cdot\|$ means size of the maximum-magnitude eigenvalue). It is also a fact that $\|M\|$ is less than the square root of the sum of the squares of the entries of M:

$$\|M\| \leq \sqrt{\sum_{j,k} |M_{j,k}|^2}.$$

What is a sufficient condition on $|\Gamma_1|$ and $|\Gamma_2|$ such that $\Phi_\Gamma^* \Phi_\Gamma$ is invertible?

(d) Complete this sentence:
No $x \in \mathbb{R}^n$ can simultaneously have $\Psi_1^* x$ non-zero only on Γ_1 and $\Psi_2^* x$ non-zero only on Γ_2 when

$$\underline{\hspace{3cm}} \quad \leq \quad \underline{\hspace{3cm}}$$

(e) Let α_1 be a vector with non-zero components only on Γ_1 and α_2 be a vector with non-zero components only on Γ_2. Set

$$f = \Phi\alpha = \begin{bmatrix} \Psi^1 & \Psi^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

Complete this sentence:
It is impossible to find a vector $\beta \in \mathbb{R}^{2n}$ that has fewer non-zero terms than α and $\Phi\beta = f$ when

$$\underline{\hspace{3cm}} \quad \leq \quad \underline{\hspace{3cm}}$$