

Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier.

– Gilbert Strang

Linear signal spaces (vector spaces)

A *vector space* is simply a collection of things that obeys certain abstract (but mostly familiar) algebraic properties. We will start by detailing these properties.

- A vector space \mathcal{S} is composed of a set of elements, called *vectors*, and members of a field¹ \mathbb{F} called *scalars*.
- The space also has rules for adding vectors and multiplying them by scalars
 - *vector addition*, which we will write as ‘+’ combines two vectors to get a third
 - *scalar multiplication*, combines a scalar and a vector to get another vector
- The ‘+’ operation must obey the following four rules for all $\mathbf{x}, \mathbf{y} \in \mathcal{S}$:
 1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutative)
 2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ (associative)
 3. There is a unique *zero vector* $\mathbf{0}$ such that

$$\mathbf{x} + \mathbf{0} = \mathbf{x} \quad \forall \mathbf{x} \in \mathcal{S}$$

¹A field is simply a set of numbers for which multiplication and addition are defined, and distribute/associate in the same manner as the reals.

4. For each vector $\mathbf{x} \in \mathcal{S}$, there is a unique vector (called $-\mathbf{x}$) such that

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$$

- Scalar multiplication must obey the following four rules for all $a, b \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{S}$:

1. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$
 $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$ (distributive)

2. $(ab)\mathbf{x} = a(b\mathbf{x})$ (associative)

3. For the multiplicative identity of \mathbb{F} , which we write as 1, we have

$$1\mathbf{x} = \mathbf{x} \quad \forall \mathbf{x} \in \mathcal{S}$$

4. For the additive identity of \mathbb{F} , which we write as 0, we have

$$0\mathbf{x} = \mathbf{0}$$

(that's the scalar zero on the left, and the vector zero on the right).

- \mathcal{S} is closed under scalar multiplication and vector addition:

$$\mathbf{x}, \mathbf{y} \in \mathcal{S} \Rightarrow a\mathbf{x} + b\mathbf{y} \in \mathcal{S}, \quad \forall a, b \in \mathbb{F}.$$

This last point is really the most salient piece of algebraic structure. In light of it, we will often use the more descriptive terminology **linear vector space**.

Examples of vector spaces

1. \mathbb{R}^N

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad \text{where the } x_i \text{ are real}$$

and we use the standard rules for vector addition and scalar multiplication

2. \mathbb{C}^N , same as above, except the x_i are complex

3. Bounded, continuous functions $f(t)$ on the interval $[a, b]$ that are real valued.

Vector addition = adding functions pointwise,
scalar multiplication = multiplying by $a \in \mathbb{R}$ pointwise,
it should be easy to see that adding two bounded, continuous functions gives you another bounded and continuous function.

4. $GF(2)^N$

Here, the scalar field is $\{0, 1\}$, and so vectors are lists of N bits. Addition for the field is modulo 2, so

$$\begin{aligned} 0 + 0 &= 0 \\ 0 + 1 &= 1 + 0 = 1 \\ 1 + 1 &= 0 \end{aligned}$$

For example,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

This space is super useful in information/coding theory

Here is an example of something which is not a vector space:

5. Bounded, continuous functions $f(t)$ on $[a, b]$ such that

$$|f(t)| \leq 2.$$

Why is this not a linear vector space?

Linear subspaces

A (non-empty) subset \mathcal{T} of \mathcal{S} is called a **linear subspace** of \mathcal{S} if

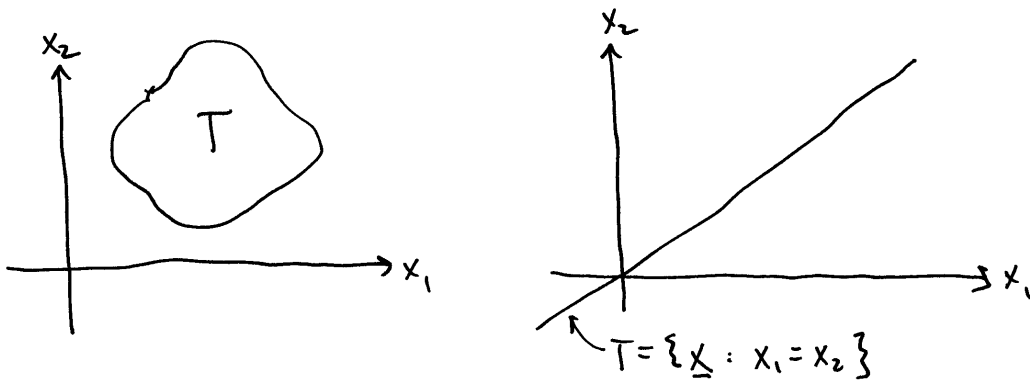
$$\forall a, b \in \mathbb{F}, \quad \mathbf{x}, \mathbf{y} \in \mathcal{T} \Rightarrow a\mathbf{x} + b\mathbf{y} \in \mathcal{T}$$

Note that it has to be true that

$$\mathbf{0} \in \mathcal{T}.$$

It is easy to show that \mathcal{T} can be treated as a linear vector space by itself.

Easy examples: Are these subspaces of $\mathcal{S} = \mathbb{R}^2$?



Which of these are subspaces?

1. $\mathcal{S} = \mathbb{R}^5$

$$\mathcal{T} = \{\mathbf{x} : x_4 = 0, x_5 = 0\}$$

2. $\mathcal{S} = \mathbb{R}^5$

$$\mathcal{T} = \{\mathbf{x} : x_4 = 1, x_5 = 1\}$$

3. $\mathcal{S} = \mathcal{C}([0, 1])$ (bounded, continuous functions on $[0, 1]$)

$$\mathcal{T} = \{\text{polynomials of degree } p\}$$

4. $\mathcal{S} =$ continuous functions on the real line

$$\mathcal{T} = \{f(t) : f \text{ is bandlimited to } \Omega\}$$

5. $\mathcal{S} = \mathbb{R}^N$

$$\mathcal{T} = \{\mathbf{x} : \mathbf{x} \text{ has no more than 5 non-zero components}\}$$

6. $\mathcal{S} = \mathbb{R}^N$

$$\mathcal{T} = \{\mathbf{x} : \mathbf{c}^T \mathbf{x} = 3\}, \text{ where } \mathbf{c} \in \mathbb{R}^N \text{ is a fixed vector}$$

(Recall the standard dot product $\mathbf{c}^T \mathbf{x} = \sum_{n=1}^N c[n]x[n]$)

7. $\mathcal{S} = \mathcal{C}([0, 1])$

$$\mathcal{T} = \{f(t) : f(t) = a \cos(2\pi t) + b \sin(2\pi t) \text{ for some } a, b \in \mathbb{R}\}$$

Linear combinations and spans

Let $\mathcal{M} = \{\mathbf{v}_1, \dots, \mathbf{v}_N\}$ be a set of vectors in a linear space \mathcal{S} .

Definition: A **linear combination** of vectors in \mathcal{M} is a sum of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_N\mathbf{v}_N$$

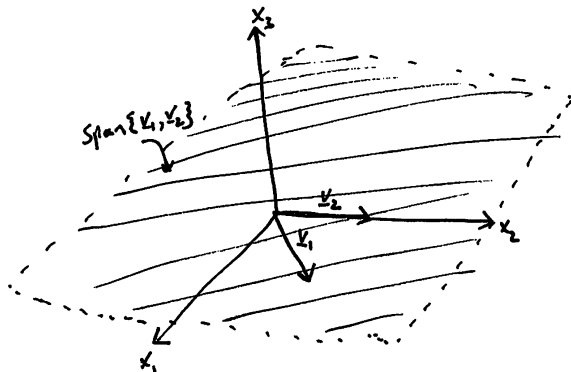
for some $a_1, \dots, a_N \in \mathbb{F}$.

Definition: The **span** of \mathcal{M} is the set of all linear combinations of \mathcal{M} . We write this as

$$\text{span}(\mathcal{M}) = \text{span}(\{\mathbf{v}_1, \dots, \mathbf{v}_N\})$$

Example:

$$\mathcal{S} = \mathbb{R}^3, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



$\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\}) = (x_1, x_2)$ plane

i.e. for any x_1, x_2 we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$

Question: What is the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad ?$$

What about if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad ?$$

Example:

$$\begin{aligned} \mathcal{S} &= \{x(t) : x(t) \text{ is periodic with period } 2\pi\} \\ \mathcal{M} &= \{e^{jkt}\}_{k=-B}^B \end{aligned}$$

Then $\text{span}(\mathcal{M}) =$ periodic, bandlimited (to B) functions, i.e.

$$x(t) = \sum_{k=-B}^B c_k e^{jkt}$$

for some $c_{-B}, c_{-B+1}, \dots, c_0, c_1, \dots, c_B \in \mathbb{C}$.

Linear dependence

A set of vectors $\{\mathbf{v}_j\}_{j=1}^N$ is said to be **linearly dependent** if there exists scalars a_1, \dots, a_N , not all $= 0$, such that

$$\sum_{n=1}^N a_n \mathbf{v}_n = \mathbf{0}$$

Likewise, if $\sum_n a_n \mathbf{v}_n = \mathbf{0}$ only when all the $a_j = 0$, then $\{\mathbf{v}_n\}_{n=1}^N$ is said to be **linearly independent**.

Example 1:

$$\mathcal{S} = \mathbb{R}^3, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Find a_1, a_2, a_3 such that

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 = \mathbf{0}$$

Note that any two of the vectors above are linearly independent:

$$\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2\}) = \text{span}(\{\mathbf{v}_1, \mathbf{v}_3\}) = \text{span}(\{\mathbf{v}_2, \mathbf{v}_3\})$$

Example 2:

$$\begin{aligned}\mathcal{S} &= \mathcal{C}([0, 1]) \\ \mathbf{v}_1 &= \cos(2\pi t) \\ \mathbf{v}_2 &= \sin(2\pi t) \\ \mathbf{v}_3 &= 2 \cos(2\pi t + \pi/3)\end{aligned}$$

Find a_1, a_2, a_3 such that

$$a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 = \mathbf{0}$$

Repeat for

$$\mathbf{v}_3 = A \cos(2\pi t + \phi) \quad \text{for some } A > 0, \quad \phi \in [0, 2\pi).$$

Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ are linearly dependent. Then

$$\sum_n a_n \mathbf{v}_n = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_k = \frac{1}{a_k} \sum_{n \neq k} a_n \mathbf{v}_n \quad \text{for every } a_k \neq 0.$$

Thus there is at least one vector we can remove from the set without changing its span. This process can be repeated until we are left with a set that is linearly independent.

Bases

Definition: A **basis** of a linear vector space \mathcal{S} is a (countable) set of vectors \mathcal{B} such that²

1. $\text{span}(\mathcal{B}) = \mathcal{S}$
2. \mathcal{B} is linearly independent

The second condition ensures that all bases of \mathcal{S} will have the same (possibly infinite) number of elements.

The **dimension** of \mathcal{S} is the number of elements required in a basis for \mathcal{S} . (Again, this could very easily be ∞ .)

Examples:

1. \mathbb{R}^N with

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

This is the **standard basis** for \mathbb{R}^N .

2. \mathbb{R}^N with any set of N linearly independent vectors.

²In infinite dimensions, we really need to be more careful with this definition than what is being said here. In that setting, there are multiple definitions of a basis, the most useful of which require the notion of an inner product, which we will get to soon. We will return to this technical issue then.

3. $\mathcal{S} = \{\text{polynomials of degree at most } p\}$.
 A basis for \mathcal{S} is $\mathcal{B} = \{1, t, t^2, \dots, t^p\}$.
 The dimension of \mathcal{S} is $p + 1$.
4. $\mathcal{S} = \{x(t) : x(t) \text{ is periodic with period } 2\pi\}$
 A basis for \mathcal{S} is $\mathcal{B} = \{e^{jkt}\}_{k=-\infty}^{\infty}$ (Fourier Series)
 \mathcal{S} is infinite dimensional.
5. $\mathcal{S} = GF(2)^3$ (length 3 bit vectors with mod 2 arithmetic).
 A basis for \mathcal{S} is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

How would you write

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{\quad} \mathbf{v}_1 + \underline{\quad} \mathbf{v}_2 + \underline{\quad} \mathbf{v}_3 \quad ?$$