Parseval's Theorem

One handy fact (and a fact we have used many times in this course already) about the Fourier transform is that it is **energy preserving**,

$$\|x(t)\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 \, \mathrm{d}\Omega = \frac{1}{2\pi}\|X(j\Omega)\|_2^2,$$

and more generally, it preserves the L_2 inner product:

$$\begin{aligned}\langle x(t), y(t)\rangle &= \int_{-\infty}^{\infty} x(t)\overline{y(t)}\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\overline{Y(j\Omega)}\,\mathrm{d}\Omega \\ &= \frac{1}{2\pi}\langle X(j\Omega), Y(j\Omega)\rangle.\end{aligned}$$

It is not not too hard to show that something very similar is true for any orthobasis expansion. Let $\mathcal{S}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_S$ which induces norm $\|\cdot\|_S$. Let $\{v_k\}_{k\in\Gamma}$ be an orthobasis[1] for $\mathcal{S}$. Then for every $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$,

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_S = \sum_{k\in\Gamma} \alpha_k \overline{\beta_k},$$

where

$$\alpha_k = \langle \boldsymbol{x}, \boldsymbol{v}_k\rangle_S, \qquad \beta_k = \langle \boldsymbol{y}, \boldsymbol{v}_k\rangle_S.$$

You can think of the $\{\alpha_k\}$ as the transform coefficients of $\boldsymbol{x}$ and the $\{\beta_k\}$ as the transform coefficients of $\boldsymbol{y}$. So we have

$$\begin{aligned}\langle \boldsymbol{x}, \boldsymbol{y}\rangle_S &= \langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle_{\ell_2}, \\ \|\boldsymbol{x}\|_S^2 &= \|\boldsymbol{\alpha}\|_2^2.\end{aligned}$$

[1] We are using Γ to be an arbitrary index set here; it can be either finite, e.g. $\Gamma = 1, 2, \ldots, N$, or infinite, e.g. $\Gamma = \mathbb{Z}$.

$\Rightarrow$ An orthobasis makes every Hilbert space **equivalent** to ℓ_2.

All of the geometry (lengths, angles) maps into standard Euclidean geometry in coefficient space. As you can imagine, this is a pretty useful fact.

Proof of Parseval. With $\alpha_k = \langle \boldsymbol{x}, \boldsymbol{v}_k \rangle$ and $\beta_k = \langle \boldsymbol{y}, \boldsymbol{v}_k \rangle$, we can write

$$\boldsymbol{x} = \sum_{k\in\Gamma} \alpha_k \, \boldsymbol{v}_k, \quad \text{and} \quad \boldsymbol{y} = \sum_{k\in\Gamma} \beta_k \, \boldsymbol{v}_k,$$

and so

$$\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{y} \rangle_S &= \left\langle \sum_{k\in\Gamma} \alpha_k \boldsymbol{v}_k, \sum_{\ell\in\Gamma} \beta_\ell \boldsymbol{v}_\ell \right\rangle_S \\
&= \sum_{k\in\Gamma} \alpha_k \left\langle \boldsymbol{v}_k, \sum_{\ell\in\Gamma} \beta_\ell \boldsymbol{v}_\ell \right\rangle_S \\
&= \sum_{k\in\Gamma} \sum_{\ell\in\Gamma} \alpha_k \overline{\beta_\ell} \langle \boldsymbol{v}_k, \boldsymbol{v}_\ell \rangle_S.
\end{aligned}$$

For a fixed value of k, only one term in the inner sum above will be nonzero, as $\langle \boldsymbol{v}_k, \boldsymbol{v}_\ell \rangle = 0$ unless $\ell = k$. Thus

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_S = \sum_{k\in\Gamma} \alpha_k \overline{\beta_k}.$$

A straightforward consequence of the result above is that distances in $\mathcal{S}$ under the induced norm are equivalent to Euclidean (ℓ_2) distances in coefficient space:

$$\|\boldsymbol{x}-\boldsymbol{y}\|_S = \|\boldsymbol{\alpha}-\boldsymbol{\beta}\|_2 = \left(\sum_{k\in\Gamma}(\alpha_k-\beta_k)^2\right)^{1/2}.$$

Thus changing the value of an orthobasis expansion coefficient by an amount ϵ will change the signal by an amount (as measured in $\|\cdot\|_S$) ϵ.

To be more precise about this, suppose $\boldsymbol{x}$ has transform coefficients $\{\alpha_k = \langle \boldsymbol{x}, \boldsymbol{v}_k\rangle_S\}$. If I perturb one of them, say at location k_0, by setting

$$\tilde{\alpha}_k = \begin{cases} \alpha_{k_0}+\epsilon & k=k_0 \\ \alpha_k & k\neq k_0 \end{cases},$$

and then synthesizing

$$\tilde{\boldsymbol{x}} = \sum_{k\in\Gamma}\tilde{\alpha}_k\boldsymbol{v}_k,$$

we will have

$$\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|_S = \epsilon.$$

Notice that while the error is localized to one expansion coefficient, it could effect the entire reconstruction, but its net effect will still be ϵ.

Here is another example. Suppose I sample a signal $x_c(t)$ which is bandlimited to π/T at a rate T, producing the sample sequence $x[n] = x_c(nT)$. Each of these samples gets perturbed by a (possibly different) amount $\epsilon[n]$:

$$\tilde{x}[n] = x[n] + \epsilon[n].$$

We resynthesize the signal using sinc interpolation:

$$\tilde{x}_c(t) = \sum_{n=-\infty}^{\infty} \tilde{x}[n] h_T(t - nT),$$

and the difference between this signal and the "true" signal is

$$\begin{aligned} x_c(t) - \tilde{x}_c(t) &= \sum_{n=-\infty}^{\infty} (x[n] - \tilde{x}[n]) h_T(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \sqrt{T}(x[n] - \tilde{x}[n])\, h_T(t - nT)/\sqrt{T}. \end{aligned}$$

Since the $\{h_T(t - nT)/\sqrt{T}\}_{n\in\mathbb{Z}}$ are an orthobasis for $B_{\pi/T}$, we know

$$\begin{aligned} \|x_c(t) - \tilde{x}_c(t)\|_{L_2}^2 &= \int |x_c(t) - \tilde{x}_c(t)|^2 \ \mathrm{d}t \\ &= \sum_{n=-\infty}^{\infty} \left|\sqrt{T}(x[n] - \tilde{x}[n])\right|^2 \\ &= T \sum_{n=-\infty}^{\infty} |\epsilon[n]|^2 \end{aligned}$$

The upshot of this is that as we change each sample, we know exactly what the net effect will be on the reconstruction error.

Truncating ortho expansions and linear approximation

Say $\{\boldsymbol{v}_k\}_{k=0}^{\infty}$ in and orthobasis for a Hilbert space $\mathcal{S}$. Let $\mathcal{T}$ be the subspace spanned by the first 10 elements of $\{\boldsymbol{v}_k\}$:

$$\mathcal{T} = \operatorname{span}\left(\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_9\}\right).$$

1. Given $\boldsymbol{x} \in \mathcal{S}$, what is the closest point in $\mathcal{T}$ (call it $\hat{\boldsymbol{x}}$) to $\boldsymbol{x}$? We have seen that it is

$$\hat{\boldsymbol{x}} = \sum_{k=0}^{9} \langle \boldsymbol{x}, \boldsymbol{v}_k \rangle \boldsymbol{v}_k.$$

2. How good an approximation is $\hat{\boldsymbol{x}}$ to $\boldsymbol{x}$? If we measure this in the induced norm $\|\cdot\|_S$, then

$$\begin{aligned}\|\boldsymbol{x} - \hat{\boldsymbol{x}}\|_S^2 &= \left\| \sum_{k=0}^{\infty} \langle \boldsymbol{x}, \boldsymbol{v}_k \rangle \boldsymbol{v}_k - \sum_{k=0}^{9} \langle \boldsymbol{x}, \boldsymbol{v}_k \rangle \boldsymbol{v}_k \right\|_S^2 \\ &= \left\| \sum_{k=10}^{\infty} \langle \boldsymbol{x}, \boldsymbol{v}_k \rangle \boldsymbol{v}_k \right\|_S^2 \\ &= \sum_{k=10}^{\infty} |\langle \boldsymbol{x}, \boldsymbol{v}_k \rangle|^2.\end{aligned}$$

Since also

$$\|\boldsymbol{x}\|_S^2 = \sum_{k=0}^{\infty} |\langle \boldsymbol{x}, \boldsymbol{v}_k \rangle|^2$$

the approximation error for $\hat{x}$ will be small if the first 10 transform coefficients

$$\langle \boldsymbol{x}, \boldsymbol{v}_0 \rangle,\ \langle \boldsymbol{x}, \boldsymbol{v}_1 \rangle,\ \ldots,\ \langle \boldsymbol{x}, \boldsymbol{v}_9 \rangle,$$

contain "most" of the total energy.

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Of course, there is nothing special about taking the first 10 coefficients. We can just as easily form a K term approximation using

$$\hat{\boldsymbol{x}}_K = \sum_{k=0}^{K-1} \langle \boldsymbol{x}, \boldsymbol{v}_k \rangle \boldsymbol{v}_k$$

which has error

$$\|\boldsymbol{x} - \hat{\boldsymbol{x}}_K\|_S^2 = \sum_{k=K}^{\infty} |\langle \boldsymbol{x}, \boldsymbol{v}_k \rangle|^2.$$

If the sum above is small for moderately large K, we can "compress" $\boldsymbol{x}$ by using just the first K terms in the expansion.

This is precisely what is done in image and video compression — more details on this in a couple of lectures.

Example:
Any real-valued function on $[-1/2, 1/2]$ with even symmetry can be built up out of harmonic cosines:

$$x(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \sqrt{2} \cos(2\pi k t).$$

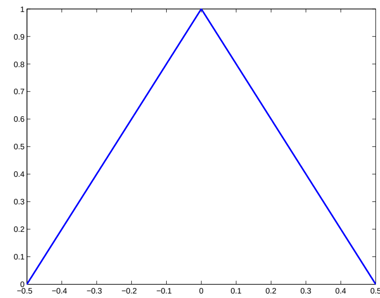
(That this is true follows directly from the observation that every signal on $[-1/2, 1/2]$ that is real-valued and even has a Fourier series which is real-valued and even.) This is an orthobasis expansion in the standard inner product with

$$v_0(t) = 1,\ v_1(t) = \sqrt{2}\cos(2\pi t),\ \ldots,\ v_k(t) = \sqrt{2}\cos(2\pi k t),\ \ldots$$

It is easy to check that $\langle \boldsymbol{v}_k, \boldsymbol{v}_\ell \rangle = 0,\ k \neq \ell$ and $\langle \boldsymbol{v}_k, \boldsymbol{v}_k \rangle = 1$.

For the triangle function below

$$x(t) = \begin{cases} 1+2t, & -1/2 \le t \le 0 \\ 1-2t, & 0 \le t \le 1/2 \end{cases}$$



the expansion coefficients are

$$\begin{aligned} \alpha_0 &= 1/2, \\ \alpha_k &= \int_{-1/2}^{1/2} x(t)\sqrt{2}\cos(2\pi k t)\,\mathrm{d}t \\ &= 2\sqrt{2}\int_0^{1/2} (1-2t)\cos(2\pi k t)\,\mathrm{d}t \\ &= \begin{cases} 0 & k \text{ even, } k \neq 0 \\ \frac{2\sqrt{2}}{\pi^2 k^2} & k \text{ odd} \end{cases}. \end{aligned}$$

First, let's compute the norm in time and coefficient space just to make sure they agree:

$$\|\boldsymbol{x}\|_2^2 = \int_{-1/2}^{1/2} |x(t)|^2\,\mathrm{d}t = 2\int_0^{1/2} (1-2t)^2\,\mathrm{d}t = 1/3,$$

and

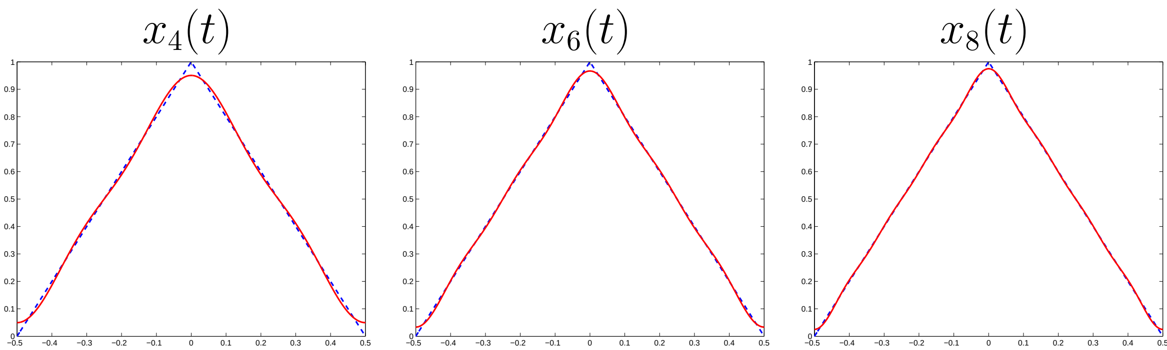
$$\begin{aligned} \sum_{k=0}^{\infty} |\alpha_k|^2 &= \frac{1}{4} + \frac{8}{\pi^4}\sum_{k'=0}^{\infty}\frac{1}{(1+2k')^4} \\ &= \frac{1}{4} + \frac{8}{\pi^4}\left(\frac{\pi^4}{96}\right) \\ &= \frac{1}{3}. \end{aligned}$$

When we truncate the expansion at K terms,

$$x_K(t) = \frac{1}{2} + \sum_{k=1}^{K-1} \alpha_k \sqrt{2}\cos(2\pi k t),$$

we can interpret the result as an **approximation** of $x(t)$ that is a member of the K-dimensional subspace $\mathrm{span}(\{\sqrt{2}\cos(2\pi k t\}_{k=0}^{K-1})$, and we know that it is the best approximation in that subspace.

Here are the approximation for $K = 4, 6, 8$:



We can compute the error in each of these approximations explicitly, as

$$\begin{aligned} x(t) - x_K(t) &= \sum_{k=0}^{\infty} \alpha_k \sqrt{2}\cos(2\pi k t) - \sum_{k=0}^{K-1} \alpha_k \sqrt{2}\cos(2\pi k t) \\ &= \sum_{k=K}^{\infty} \alpha_k \sqrt{2}\cos(2\pi k t), \end{aligned}$$

and so

$$\|x(t) - x_K(t)\|_2^2 = \sum_{k=K}^{\infty} |\alpha_k|^2,$$

or, since $x_K(t) \perp x(t) - x_K(t)$,

$$\|x(t) - x_K(t)\|_2^2 = \|x(t)\|_2^2 - \|x_K(t)\|_2^2.$$

In the three examples above, we have

$$\|x(t) - x_4(t)\|_2^2 \approx 1.92 \cdot 10^{-4}, \quad \|x(t) - x_6(t)\|_2^2 \approx 6.01 \cdot 10^{-5},$$
$$\|x(t) - x_8(t)\|_2^2 \approx 2.59 \cdot 10^{-5}.$$

The Gram-Schmidt algorithm

We have seen that orthobases for a Hilbert space (or a subspace) have many nice properties. Given any basis $\{\boldsymbol{v}_n\}_{n=1}^N$ for an N-dimensional space (or subspace), we can turn it into and orthobasis using the **Gram-Schmidt algorithm**.

The goal is to take a sequence of signals $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$ and produce $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N\}$ such that

$$\operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}) = \operatorname{span}(\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\})$$

and

$$\langle \boldsymbol{u}_n, \boldsymbol{u}_\ell \rangle = \begin{cases} 1 & n = \ell, \\ 0 & n \neq \ell \end{cases}.$$

That is $\{\boldsymbol{u}_n\}$ spans the same space as $\{\boldsymbol{v}_n\}$, but it is a orthobasis.

1. Choose $\boldsymbol{w}_1 = \boldsymbol{v}_1$ and normalize it to get[2]

   $$\boldsymbol{u}_1 = \frac{\boldsymbol{w}_1}{\|\boldsymbol{w}_1\|}.$$

   Clearly, $\boldsymbol{u}_1$ is an orthobasis for span$(\{\boldsymbol{v}_1\})$.

2. To get $\boldsymbol{u}_2$, we subtract from $\boldsymbol{v}_2$ its projection onto $\boldsymbol{u}_1$:

   $$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \langle \boldsymbol{v}_2, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1$$
   $$\boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|}$$

[2]The norm here and below is the one induced by the inner product.

Note that $\boldsymbol{u}_2$ is orthogonal to $\boldsymbol{u}_1$ by the orthogonality principle, but just to make sure

$$\begin{aligned}\langle \boldsymbol{u}_2, \boldsymbol{u}_1 \rangle &= \frac{1}{\|\boldsymbol{w}_2\|}\langle \boldsymbol{w}_2, \boldsymbol{u}_1 \rangle \\ &= \frac{1}{\|\boldsymbol{w}_2\|}\left(\langle \boldsymbol{v}_2, \boldsymbol{u}_1 \rangle - \langle \boldsymbol{v}_2, \boldsymbol{u}_1 \rangle \langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle\right) \\ &= 0.\end{aligned}$$

So $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is an orthobasis for $\mathrm{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$.

3. At the beginning of the kth step, $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{k-1}\}$ is an orthobasis for $\mathrm{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\})$. We get $\boldsymbol{u}_k$ by subtracting off its projection onto $\mathrm{span}(\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{k-1}\})$ and normalizing:

$$\begin{aligned}\boldsymbol{w}_k &= \boldsymbol{v}_k - \sum_{\ell=1}^{k-1} \langle \boldsymbol{v}_k, \boldsymbol{u}_\ell \rangle \boldsymbol{u}_\ell, \\ \boldsymbol{u}_k &= \frac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|}.\end{aligned}$$

By induction, $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ is and orthobasis for $\mathrm{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\})$.

Note: If at any point

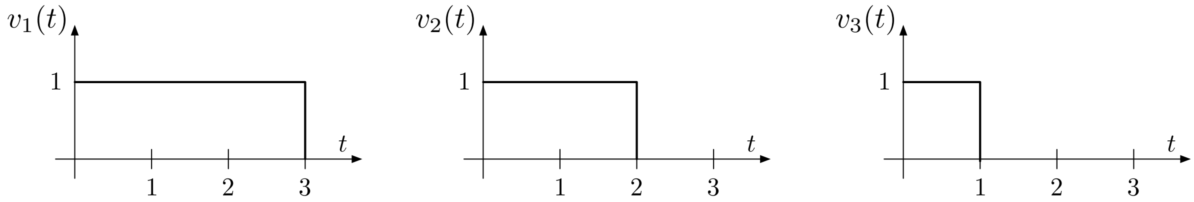
$$\boldsymbol{v}_k \in \mathrm{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\})$$

(which means the $\{\boldsymbol{v}_n\}$ are linearly dependent — and not a basis), we will have

$$\boldsymbol{u}_k = \boldsymbol{0}.$$

When this happens, we can simply throw away $\boldsymbol{u}_k, \boldsymbol{v}_k$ and move on. The set of $\{\boldsymbol{u}_k\}$ will be smaller than N, but will still be an orthobasis for $\mathrm{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\})$.

Exercise: Let $\mathcal{S}$ be the space of piecewise-constant signals on $[0,1), [1,2), [2,3]$ with the standard L_2 inner product. Turn the following basis



into an orthobasis using Gram-Schmidt.