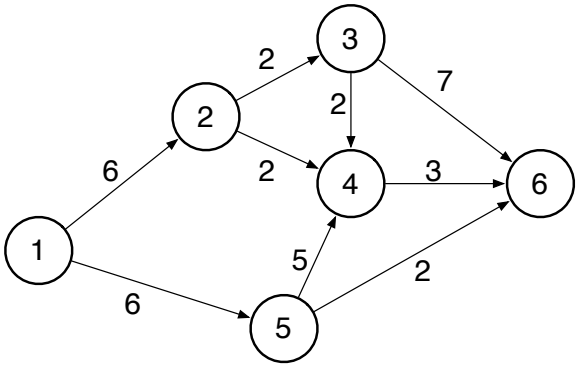
Max flow, Min cut

Consider the following network.



The nodes are routers, the edges are communications links; associated with each node is a capacity — node 1 can communicate to node 2 at as much as 6 Mbps, node 2 can communicate to node 4 at upto 2 Mbps, etc.

Question: Can node 1 (the source) communicate to node 6 (the sink) at 6 Mbps? 12 Mbps? What is the maximum rate?

Formally, we can model this type of problem as follows. The connections between the N nodes in the network are entries in an $N \times N$ matrix; entry (i, j) of the capacity matrix $\boldsymbol{C}$ records the capacity of link from node i to node j. If there is not a link from i to j, we set $C(i, j) = 0$. Here is the capacity matrix for the example network above:

$$\boldsymbol{C} = \begin{bmatrix} 0 & 6 & 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We will solve for the $N \times N$ flow matrix $\boldsymbol{X}$:

$$X[i,j] = \text{flow from node } i \text{ to node } j.$$

Valid flows are positive and less than their respective capacities:

$$\boldsymbol{X} \geq \boldsymbol{0}, \quad \boldsymbol{X} \leq \boldsymbol{C}.$$

(The inequality above are to be understood entrywise, as the partial ordering for the non-negative orthant $\mathbb{R}_+^{N\times N}$.) To keep things simple, we will assume that there are no edges coming into the source or out of the sink. The *conservation of flow* means that the sum of all the outgoing flows at every non-source/sink node must be the same as the sum of all the incoming flows. The sum of all the outgoing flows from node i is simply the sum of all entries in row i of $\boldsymbol{X}$; the sum of all the incoming flows is the sum along column i. We will assume the source is node $i = 1$ and the sink is node $i = N$, so the conservation law becomes the $N - 2$ linear equality constraints

$$\sum_{j=2}^{N} X(i,j) = \sum_{k=1}^{N-1} X(k,i), \quad i = 2, \ldots, N-1.$$

The flow of the network, then, is simply the sum of everything coming out of the source:

$$\text{flow} = \sum_{i=2}^{N} X(1,i)$$

So, solving for the **maximum flow is a linear program**:

$$\begin{aligned} \underset{\boldsymbol{X}\in\mathbb{R}^{N\times N}}{\text{maximize}} \ \langle \boldsymbol{X}, \boldsymbol{S} \rangle \quad \text{subject to} \quad & -\boldsymbol{X} \leq \boldsymbol{0} \\ & \boldsymbol{X} \leq \boldsymbol{C} \\ & \langle \boldsymbol{X}, \boldsymbol{L}_n \rangle = \boldsymbol{0}, \quad n = 2, \ldots, N-1 \end{aligned}$$

where

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{L}_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and similarly $\boldsymbol{L}_n$ consists of a single column (n) of ones (except for the last row) minus a single row (also n) of ones (except for the first column).

A general LP with both linear inequality and equality constraints

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \ \langle \boldsymbol{x}, \boldsymbol{c} \rangle, \quad \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{W}\boldsymbol{x} = \boldsymbol{y}, \end{aligned}$$

has dual

$$\begin{aligned} \underset{\boldsymbol{\lambda}, \boldsymbol{\nu}}{\text{maximize}} \ -\langle \boldsymbol{\lambda}, \boldsymbol{b} \rangle - \langle \boldsymbol{\nu}, \boldsymbol{y} \rangle \quad \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda} + \boldsymbol{W}^{\mathrm{T}}\boldsymbol{\nu} + \boldsymbol{c} = \boldsymbol{0} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned}$$

A quick calculation shows that the dual of maxflow is then

$$\begin{aligned} \underset{\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \boldsymbol{\nu}}{\text{minimize}} \ \langle \boldsymbol{\Lambda}_1, \boldsymbol{C} \rangle \quad \text{subject to} \quad & \boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2 + \boldsymbol{Q} - \boldsymbol{S} = \boldsymbol{0} \\ & \boldsymbol{\Lambda}_1 \geq \boldsymbol{0} \\ & \boldsymbol{\Lambda}_2 \geq \boldsymbol{0} \end{aligned}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} 0 & \nu_2 & \nu_3 & \cdots & \nu_{N-1} & 0 \\ 0 & 0 & \nu_3 - \nu_2 & \cdots & \nu_{N-1} - \nu_2 & -\nu_2 \\ 0 & \nu_2 - \nu_3 & 0 & \cdots & \nu_{N-1} - \nu_3 & -\nu_3 \\ \vdots & \cdots & & & & \vdots \\ 0 & \nu_2 - \nu_{N-1} & \nu_3 - \nu_{N-1} & \cdots & 0 & -\nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We see that the $\boldsymbol{\Lambda}_2$ are just slack variables, and we can re-write the program as

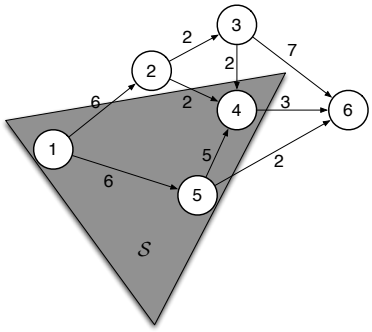
$$\underset{\boldsymbol{\Lambda},\boldsymbol{\nu}}{\text{minimize}} \ \langle \boldsymbol{\Lambda}, \boldsymbol{C} \rangle \quad \text{subject to} \quad \begin{aligned} \boldsymbol{\Lambda} + \boldsymbol{Q} &\geq \boldsymbol{S} \\ \boldsymbol{\Lambda} &\geq \boldsymbol{0}, \end{aligned}$$

or equivalently

$$\begin{aligned} \underset{\boldsymbol{\Lambda},\boldsymbol{\nu}}{\text{minimize}} \ \sum_{i,j} \lambda_{i,j} C_{i,j} \quad \text{subject to} \quad \lambda_{i,j} - \nu_j + \nu_i &\geq 0, \quad 2 \leq i,j \leq N-1 \\ \lambda_{1,j} + \nu_j &\geq 1, \quad j = 2, \ldots, N-1 \\ \lambda_{i,N} - \nu_i &\geq 0, \quad i = 2, \ldots, N-1 \\ \lambda_{1,N} &\geq 1 \\ \lambda_{i,j} &\geq 0, \quad 1 \leq i,j \leq N. \end{aligned} \tag{1}$$

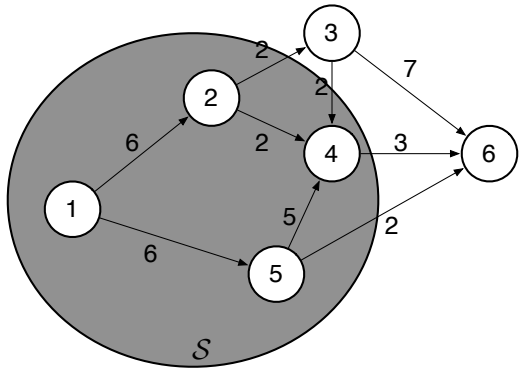
We will argue below that this dual program is equivalent to finding the **minimum cut** in the network. A cut of the network separates the vertices into two sets: one containing the source (we call this set $\mathcal{S}$, and one containing the sink. The capacity of the cut is the total value of the edges coming out of $\mathcal{S}$ — we are separating the sets by "cutting off the flow" along these edges.

For example, here we have $\mathcal{S} = \{1, 4, 5\}$:



The edges in the cut are $1 \to 2$, $4 \to 6$, and $5 \to 6$; the capacity of this cut is $6 + 3 + 2 = 11$.

In this example, we have $\mathcal{S} = \{1, 2, 4, 5\}$:



The edges in this cut are $2 \to 3$, $4 \to 6$, and $5 \to 6$. The capacity of this cut is $2 + 3 + 2 = 7$.

In general, a cut is specified by a subset of vertices containing the source but not the sink: $\mathcal{S} \subset \{1, \ldots, N\}$, $1 \in \mathcal{S}$, $N \notin \mathcal{S}$. The associated capacity is

$$\text{capacity}(\mathcal{S}) = \sum_{i \in \mathcal{S},\ j \notin \mathcal{S}} C_{i,j}$$

What is the minimum value of the smallest cut? We will argue that it is same as the optimal value of the solution $d^\star$ of the dual program in (1). First, suppose that $\mathcal{S}$ is a valid cut. From $\mathcal{S}$, we can easily find a dual feasible point that matches its capacity: for $n = 1, \ldots, N$, take

$$\nu_n = \begin{cases} 1, & n \in \mathcal{S}, \\ 0, & n \notin \mathcal{S}, \end{cases} \quad \text{and} \quad \lambda_{i,j} = \begin{cases} \max(\nu_i - \nu_j, 0), & i \neq 1,\ j \neq N, \\ 1 - \nu_j, & i = 1, \\ \nu_i, & j = N \end{cases}.$$

Notice that these choices obey the constraints in the dual, and that $\lambda_{i,j}$ will be 1 if $i \to j$ is cut, and 0 otherwise, so

$$\text{capactity}(\mathcal{S}) = \sum_{i,j} \lambda_{i,j} C_{i,j}.$$

Every cut is feasible, so

$$d^\star \leq \text{MINCUT}.$$

Now we show that for every solution $\boldsymbol{\nu}^\star, \boldsymbol{\lambda}^\star$ of the dual, there is a cut that has a capacity at most $d^\star$. The argument for this is nifty: we generate a cut *at random*, and then show that the expected value of the capacity of the cut is less than $d^\star$ — this means there must be at least one with a capacity of $d^\star$ or less.

Let Z be a uniform random variable on $[0, 1]$. Along with $\boldsymbol{\lambda}^\star, \nu_2^\star, \ldots, \nu_{N-1}^\star$ generated by solving (1), take $\nu_1 = 1$ and $\nu_N = 0$. Create a cut $\mathcal{S}$ with the rule:

$$\text{if } \nu_n^\star > Z, \text{ then take } n \in \mathcal{S}.$$

The probability that a particular edge $i \to j$ is in this cut is

$$\begin{aligned}
\mathrm{P}\left(i \in \mathcal{S}, j \notin \mathcal{S}\right) &= \mathrm{P}\left(\nu_j^\star \leq Z \leq \nu_i^\star\right) \\
&\leq \begin{cases} \max(\nu_i^\star - \nu_j^\star, 0), & 2 \leq i, j \leq N-1, \\ 1 - \nu_j^\star, & i = 1;\ j = 2, \ldots, N-1, \\ \nu_i^\star, & i = 2, \ldots, N-1;\ j = N \\ 1, & i = 1;\ j = N. \end{cases} \\
&\leq \lambda_{i,j}^\star,
\end{aligned}$$

where the last inequality follows simply from the constraints in the dual program (1). This cut is random, so its capacity is a random variables, and its expectation is

$$\begin{aligned}
\mathrm{E}[\text{capacity}(\mathcal{S})] &= \sum_{i,j} C_{i,j} \, \mathrm{P}\left(i \in \mathcal{S}, j \notin \mathcal{S}\right) \\
&\leq \sum_{i,j} C_{i,j} \lambda_{i,j}^\star \\
&= d^\star.
\end{aligned}$$

Thus there must be a cut whose capacity is at most $d^\star$. This establishes that

$$\text{MINCUT} \leq d^\star.$$

Combining these two facts of course means that

$$d^\star = \text{MINCUT} = \text{MAXFLOW} = p^\star,$$

where $p^\star$ is the solution of the primal, and equality follows from strong duality for linear programming.

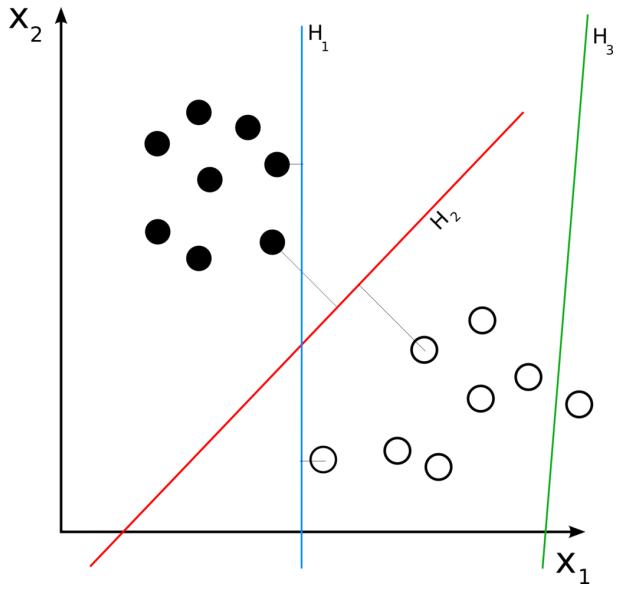
The minimum cut problem is interesting by itself. Among other things, it can be used to perform image segmentation:

(From F. Estrada et al. (2004), "Spectral embedding and min cut for image segmentation")

You set this problem up by connecting each pixel to a foreground "source" (with some capacity that would represent the foreground value) and to a background "sink" (with some capacity that would represent the background value), and penalizing if adjacent pixels get assigned to different modes. (See the reference above for all of the details.)

Support vector machines

Consider the following fundamental binary classification problem. We are given points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M \in \mathbb{R}^N$ with labels $y_1, \ldots, y_M$, where $y_m \in \{-1, +1\}$. We would like to find a hyperplane (i.e. affine functional) which *separates* the points[1]:



$\mathcal{H}_1$ and $\mathcal{H}_2$ above separate the points in $\mathbb{R}^2$, but $\mathcal{H}_3$ does not. To choose among the hyperplanes which separate the points, we will take the one with maximum margin (maximize the distance to the closest point in either class).

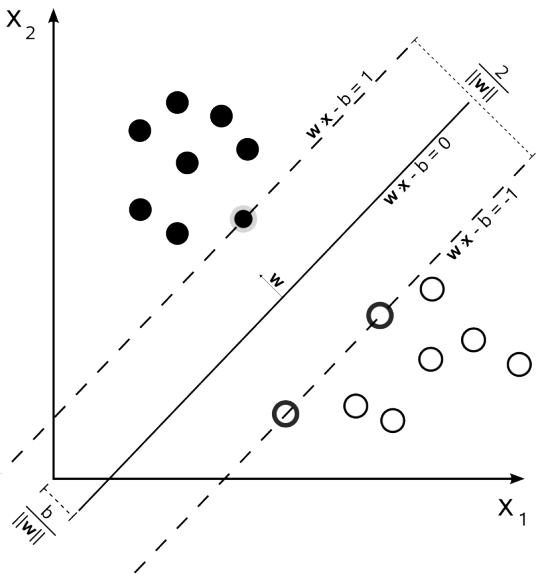
To restate this, we want to find a $\boldsymbol{w} \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that

$$\begin{aligned} \langle \boldsymbol{x}_m, \boldsymbol{w} \rangle - b &\geq 1, \quad \text{when } y_m = 1, \\ \langle \boldsymbol{x}_m, \boldsymbol{w} \rangle - b &\leq -1, \quad \text{when } y_m = -1. \end{aligned}$$

[1] From Wikipedia: "Svm separating hyperplanes (SVG)" by User:ZackWeinberg, based on PNG version by User:Cyc.

Of course, it is possible that no separating hyperplane exists; in this case, there will be no feasible points in the program above. It is straightforward, though, to modify this discussion to allow "mislabeled" points.

In the formulation above, the distance between the two (parallel) hyperplanes[2] is $2/\|\boldsymbol{w}\|_2$:



Thus maximizing this distance is the same as minimizing $\|\boldsymbol{w}\|_2$.

We have the program

$$\underset{\boldsymbol{w}\in\mathbb{R}^N,\ b\in\mathbb{R}}{\text{minimize}}\ \frac{1}{2}\|\boldsymbol{w}\|_2^2 \quad \text{subject to} \quad y_m(b-\langle \boldsymbol{x}_m, \boldsymbol{w}\rangle)+1 \leq 0,\ \ m=1,\ldots,M.$$

This is a linearly constrained quadratic program, and is clearly con-

[2]From Wikipedia: "Svm max sep hyperplane with margin" by Cyc - Own work. Licensed under Public Domain via Wikimedia Commons.

vex. The Lagrangian is

$$\begin{aligned} L(\boldsymbol{w}, b, \boldsymbol{\lambda}) &= \frac{1}{2}\|\boldsymbol{w}\|_2^2 + \sum_{m=1}^{M} \lambda_m \left[y_m (b - \langle \boldsymbol{x}_m, \boldsymbol{w} \rangle) + 1 \right] \\ &= \frac{1}{2}\|\boldsymbol{w}\|_2^2 + b\, \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{y} - \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{w} + \boldsymbol{\lambda}^{\mathrm{T}} \mathbf{1}, \end{aligned}$$

where $\boldsymbol{X}$ is the $N \times M$ matrix

$$\boldsymbol{X} = \begin{bmatrix} y_1 \boldsymbol{x}_1 & y_2 \boldsymbol{x}_2 & \cdots & y_M \boldsymbol{x}_M \end{bmatrix}.$$

The dual function is

$$g(\boldsymbol{\lambda}) = \inf_{\boldsymbol{w}, b} \left(\frac{1}{2}\|\boldsymbol{w}\|_2^2 + b\, \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{y} - \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{w} + \boldsymbol{\lambda}^{\mathrm{T}} \mathbf{1} \right).$$

Since b is unconstrained above, we see that the presence of $b\, \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{y}$ means that the dual will be $-\infty$ unless $\langle \boldsymbol{\lambda}, \boldsymbol{y} \rangle = 0$. Minimizing over $\boldsymbol{w}$, we need the gradient equal to zero,

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}, b, \boldsymbol{\lambda}) = \mathbf{0}, \quad \Rightarrow \quad \boldsymbol{w} - \boldsymbol{X}\boldsymbol{\lambda} = \mathbf{0}.$$

This means that we must have $\boldsymbol{w} = \boldsymbol{X}\boldsymbol{\lambda}$, which itself is a very handy fact as it gives us a direct passage from the dual solution to the primal solution. With these substitutions, the dual function is

$$g(\boldsymbol{\lambda}) = \begin{cases} \frac{1}{2}\|\boldsymbol{X}\boldsymbol{\lambda}\|_2^2 - \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{X}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{\lambda} + \boldsymbol{\lambda}^{\mathrm{T}} \mathbf{1}, & \langle \boldsymbol{\lambda}, \boldsymbol{y} \rangle = 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

The dual SVM program is then

$$\underset{\boldsymbol{\lambda}}{\text{maximize}} \;\; -\frac{1}{2}\|\boldsymbol{X}\boldsymbol{\lambda}\|_2^2 + \sum_{m=1}^{M} \lambda_m \quad \text{subject to} \quad \langle \boldsymbol{\lambda}, \boldsymbol{y} \rangle = 0$$
$$\boldsymbol{\lambda} \geq \mathbf{0}.$$

Given the solution $\boldsymbol{\lambda}^\star$ above, we can take $\boldsymbol{w}^\star = \boldsymbol{X}\boldsymbol{\lambda}^\star$, and the classifier is

$$\begin{aligned} f(\boldsymbol{x}) &= \langle \boldsymbol{x}, \boldsymbol{w}^\star \rangle - b^\star \\ &= \langle \boldsymbol{x}, \boldsymbol{X}\boldsymbol{\lambda}^\star \rangle - b^\star \\ &= \sum_{m=1}^{M} \lambda_m^\star y_m \langle \boldsymbol{x}, \boldsymbol{x}_m \rangle - b^\star. \end{aligned}$$

Notice that the data $\boldsymbol{x}_m$ appear only as linear functionals (i.e. inner products with) $\boldsymbol{x}$.

The key realization is that the for the dual program, the functional depends on the data $\boldsymbol{x}_m$ only through inner products, as

$$\|\boldsymbol{X}\boldsymbol{\lambda}\|_2^2 = \sum_{\ell=1}^{M} \sum_{m=1}^{M} y_\ell y_m \langle \boldsymbol{x}_\ell, \boldsymbol{x}_m \rangle.$$

This means we can replace $\langle \boldsymbol{x}_\ell, \boldsymbol{x}_m \rangle$ with any "positive kernel function" $K(\boldsymbol{x}_\ell, \boldsymbol{x}_m) : \mathbb{R}^N \otimes \mathbb{R}^N \to \mathbb{R}$ — a positive kernel just means that the $M \times M$ matrix $K(\boldsymbol{x}_\ell, \boldsymbol{x}_m)$ is in S_+^M for all choices of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_M$.

For example: you might take

$$K(\boldsymbol{x}_\ell, \boldsymbol{x}_m) = (1 + \langle \boldsymbol{x}_\ell, \boldsymbol{x}_m \rangle)^2 = 1 + 2\langle \boldsymbol{x}_\ell, \boldsymbol{x}_m \rangle + \langle \boldsymbol{x}_\ell, \boldsymbol{x}_m \rangle^2.$$

This means we have replaced the inner product of two vectors with the inner product between two vectors which have been mapped into

a higher dimensional space:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_N \\ x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \\ \sqrt{2}x_1x_2 \\ \vdots \\ \sqrt{2}x_{N-1}x_N \end{bmatrix}$$

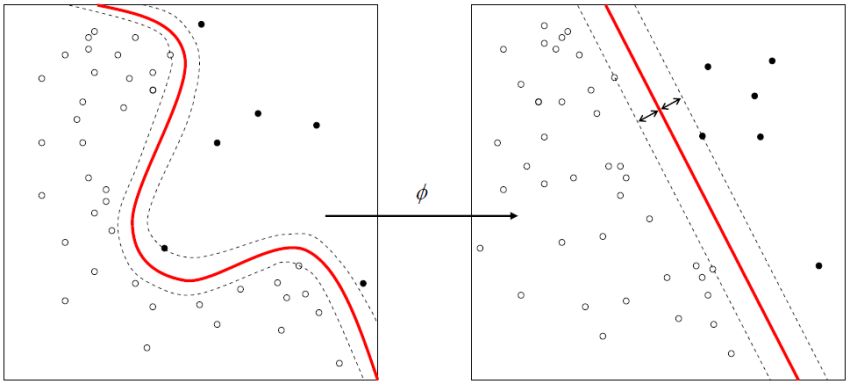
A set of linear constraints on the coordinates on the right, then, corresponds to a second order curve constraint (parabola, ellipse, hyperbola) on the coordinate on the left.

Many kernels are possible. The advantage is that to train and use the classifier, you never have to explicitly move to the higher dimensional space — you just need to be able to compute $K(\boldsymbol{x}_\ell, \boldsymbol{x}_m)$ for any pair of inputs in $\mathbb{R}^N$. A popular choice of kernel is

$$K(\boldsymbol{x}_\ell, \boldsymbol{x}_m) = \exp\left(-\gamma \|\boldsymbol{x}_\ell - \boldsymbol{x}_m\|_2^2\right).$$

This is a perfectly valid positive kernel, and it is straightforward to compute it for any pair of inputs. But it corresponds to mapping the $\boldsymbol{x}_m$ into an infinite dimensional space, then finding a hyperplane.

Here is an example from Wikipedia[3]:



[3] "Kernel Machine" by Alisneaky — Own work. Licensed under CC0 via Wikimedia Commons.