ECE 6250, Fall 2016

Homework #1

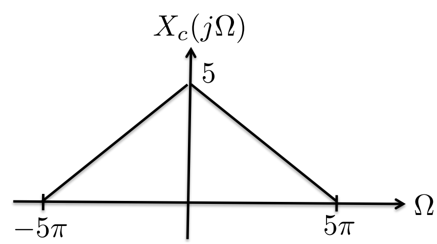
Due Thursday August 31, at the beginning of class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

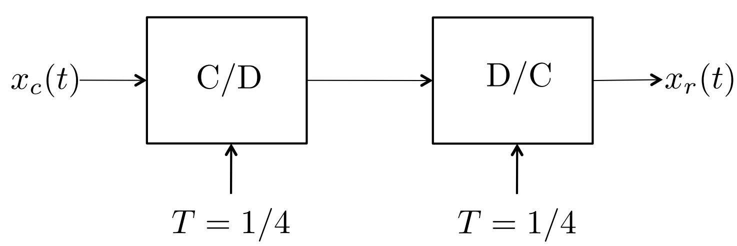
1. Sign up for Piazza if you have not already done it,
   http://piazza.com/gatech/fall2016/ece6250

2. Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

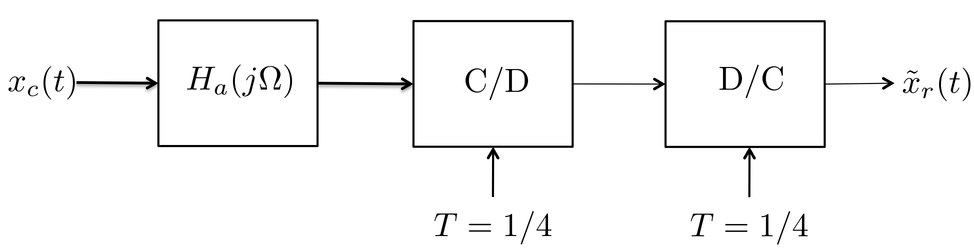
3. Suppose that a continuous-time signal $x_c(t)$ has the Fourier transform shown below:

   The signal is used as the input to two different systems, one without an anti-aliasing filter:

   and one with an anti-aliasing filter:

Above, C/D is an ideal continuous-to-discrete converter, D/C is an ideal discrete-to-continuous converter, and

$$H_a(j\Omega) = \begin{cases} 1 & |\Omega| \leq 4\pi \\ 0 & \text{else} \end{cases}.$$

(a) Calculate the energy of the reconstruction errors in both cases:

$$\|x_c(t) - x_r(t)\|_2^2 \quad \text{and} \quad \|x_c(t) - \tilde{x}_r(t)\|_2^2,$$

where

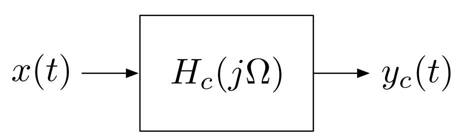
$$\|f(t)\|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

(b) Argue (rigorously) that there is no other signal that could come out of the D/C converter that is closer to $x_c(t)$ than $\tilde{x}_r(t)$. (By "closer", we mean that the energy in the error is smaller.)

Knowledge of the Parseval Theorem will definitely help for both parts (a) and (b): for continuous time signnal $x(t)$ with CTFT $X(j\Omega)$,

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \, d\Omega.$$

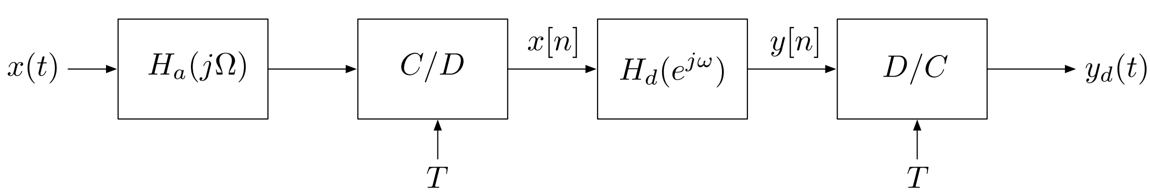
4. Consider the analog system below



where

$$H_c(j\Omega) = \frac{1}{1000\pi + j\Omega}.$$

We wish to implement a (near) equivalent system in digital using the following architecture:



where $T = 1/3000$ and $H_a(j\Omega)$ is an antialiasing filter; $H_a(j\Omega) = 1$, $|\Omega| \leq 3000\pi$ and is 0 otherwise. Find $H_d(e^{j\omega})$ such that $y_d(t)$ matches $y_c(t)$ as closely as possible. Sketch your answer.

5. Let $x_c(t)$ be a continuous-time signal which is bandlimited to π/T, and let $x[n]$ correspond to samples taken at the Nyquist rate:

$$x[n] = x_c(nT).$$

As we know, $x_c(t)$ can be perfectly reconstructed from $x[n]$ using sinc interpolation,

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}.$$

This reconstruction process can be modeled as converting $x[n]$ into a spike train, and then passing it through an ideal lowpass filter.

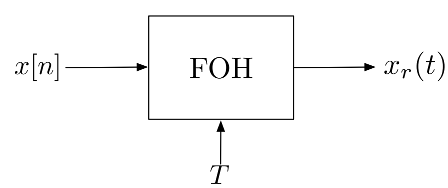
Suppose instead we generate a continuous-time signal using a different interpolation kernel:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\phi(t-nT),$$

where

$$\phi(t) = \begin{cases} 1 - \frac{|t|}{T} & -T \le t \le T \\ 0 & |t| > T \end{cases}.$$

We will use FOH to denote the system that maps $x[n]$ to $x_r(t)$:

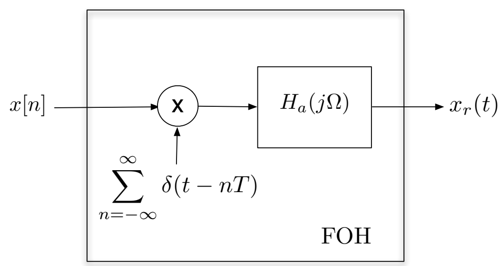


(a) Sketch $\phi(t)$. Sketch $x_r(t)$ for

$$x[n] = \delta[n+1] + 3\delta[n] - \delta[n-1] + \delta[n-2] + 2\delta[n-3],$$

where $\delta[n]$ is the Kronecker delta function ($\delta[n] = 1$ for $n = 0$ and is zero for all other values of n). How would you describe the action of FOH?

(b) We would like to model FOH as conversion to a spike train followed by an analog filter:

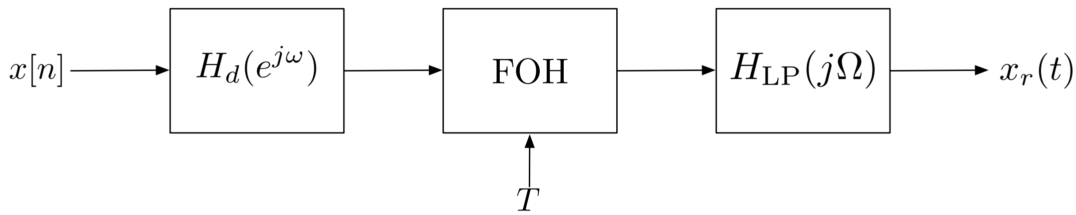


What must the frequency response $H_a(j\Omega)$ be?

(c) Now we precondition the samples $x[n]$ by passing them through a digital filter, and postcondition the output of FOH by passing it through an analog low-pass filter (see the figure below):

$$H_{\text{LP}} = \begin{cases} 1 & |\Omega| \leq \pi/T \\ 0 & \text{otherwise} \end{cases}$$

Find $H_d(e^{j\omega})$ so that $x_r(t) = x_c(t)$.



6. Let $x[n]$ be a discrete-time signal whose DTFT $X(e^{j\omega}) = 0$ for $\pi/3 \leq |\omega| \leq \pi$ which we wish to convert to analog using a standard D/C converter. Suppose that right before we feed this signal into the converter, an adversary changes exactly one sample of $x[n]$ is an unknown way at an unknown location n_0, forming

$$\tilde{x}[n] = \begin{cases} x[n] & n \neq n_0 \\ \text{something else} & n = n_0 \end{cases}.$$

(a) Given our knowledge that the original DTFT $X(e^{j\omega})$ is zero for $\pi/3 \leq |\omega| \leq \pi$, describe how such an error might be detected by examining the output $y(t)$.

(b) Describe (and sketch a diagram of) a method to correct the error. Hint: one way to do this is with C/D converters which take equally spaced samples but at different *offsets*. That is, suppose the C/D system takes samples with spacing T but offset τ, so $x[n] = x_c(\tau + nT)$.

7. Suppose that $x(t)$ is a second-order spline that defined by the overlap of 5 B-splines:

$$x(t) = \sum_{k=0}^{4} \alpha_k b_2(t - k),$$

where $b_2(t)$ is defined as on page 23 of the notes,

$$b_2(t) = \begin{cases} (t + 3/2)^2/2 & -3/2 \leq t \leq -1/2 \\ -t^2 + 3/4 & -1/2 \leq t \leq 1/2 \\ (t - 3/2)^2/2 & 1/2 \leq t \leq 3/2 \\ 0 & |t| \geq 3/2 \end{cases}$$

(a) Write a MATLAB function

```
xt = piecepoly2(t, alpha);
```

which takes $\boldsymbol{\alpha} = \{\alpha_0, \ldots, \alpha_4\}$ and returns samples of $x(t)$ at the locations specified in the vector `t`. Turn in a plot of $x(t)$ for $\boldsymbol{\alpha} = \{-1, 3, 2, -1, 4\}$. Sample it densely enough so that your plot looks like a smooth function.

(b) Suppose I tell you that

$$x(0) = 1, \quad x(1) = 1/2, \quad x(2) = -2, \quad x(3) = -3, \quad x(4) = -1.$$

What are the corresponding α_k? (Hint: you will have to construct a system of equations then solve it.)

(c) To generalize this, suppose that $x(t)$ is now a superposition of N B-splines:

$$x(t) = \sum_{n=0}^{N-1} \alpha_n b_2(t-n).$$

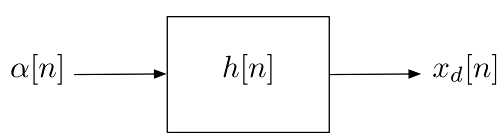
Describe how to construct the $N \times N$ matrix that maps the coefficients $\boldsymbol{\alpha}$ to the N samples $x(0), \ldots, x(N-1)$. That is, find $\boldsymbol{A}$ such that

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} & & \\ & \boldsymbol{A} & \\ & & \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

(d) To take this even further, suppose that

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha[n] b_2(t-n),$$

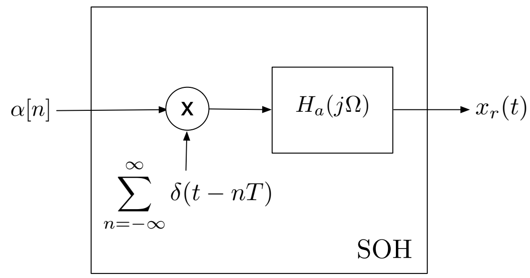
so $x(t)$ is described by the (possibly infinite) sequence of numbers $\{\alpha[n]\}_{n\in\mathbb{Z}}$. We are using slightly different notation for the expansion coefficients here since we are now going to think of $\alpha[n]$ as a discrete-time signal. Show that there discrete-time linear time-invariant system that maps $\alpha[n]$ to $x_d[n] = x(n)$ and specify its impulse response. That is, find $h[n]$ such that $x_d[n] = h[n] * \alpha[n]$.



(e) A second-order hold system takes a sequence of numbers $\alpha[n]$ and produces a continuous-time signal $x_r(t)$ that is a second-order spline:

$$x_r(t) = \sum_{n=-\infty}^{\infty} \alpha[n] g_T(t - nT), \quad g_T(t) = b_2(t/T). \tag{1}$$

(So g_T is just a "stretched" version of $b_2(t)$ that is non-zero on $[-3T/2, 3T/2]$.) This can be accomplished with a *second-order hold system*,



where now the analog filter has frequency response

$$H_a(j\Omega) = \frac{8\sin^3(\Omega T/2)}{\Omega^3}.$$

Suppose $x[n] = x_c(nT)$, where $x_c(t)$ is a second-order spline generated from $\{g_T(t - nT)\}$ as in (1) above. Find $H_d(e^{j\omega})$ in the system below so that $x_r(t) = x_c(t)$.