ECE 6250, Fall 2016

Homework #2

Due Wednesday September 7, in class

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

1. Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

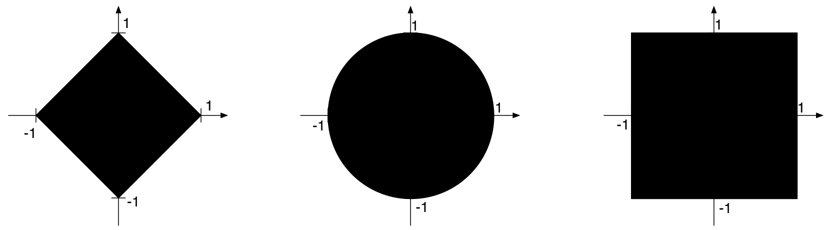
2. Prove the "reverse triangle inequality": show that in a normed linear space
$$\|\boldsymbol{x}\| - \|\boldsymbol{y}\| \ \leq \ \|\boldsymbol{x} - \boldsymbol{y}\|.$$
What can we say about $|\ \|\boldsymbol{x}\| - \|\boldsymbol{y}\|\ |$ in relation to $\|\boldsymbol{x} - \boldsymbol{y}\|$?

3. Let $\|\cdot\|_p$ be the ℓ_p norm for vectors in $\mathbb{R}^N$ as defined on page I.47 of the notes.
   (a) Prove that for any integer $p \geq 1$
   $$\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_1 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^N$$
   (b) Prove that for *any* $1 \leq q \leq p \leq \infty$
   $$\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_q \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^N.$$
   Hint: It is a fact that if a_1 and a_2 are non-negative real numbers, then
   $$(a_1 + a_2)^\alpha \geq a_1^\alpha + a_2^\alpha \quad \text{for } \alpha \geq 1.$$
   (c) *Optional bonus question*: Prove the fact in the hint above.

4. One way to visualize a norm in $\mathbb{R}^2$ is by its *unit ball*, the set of all vectors such that $\|\boldsymbol{x}\| \leq 1$. For example, we have seen that the unit balls for the ℓ_1, ℓ_2, and ℓ_∞ norms look like:

   Given an appropriate subset of the plane, $B \subset \mathbb{R}^2$, it might be possible to define a corresponding norm using
   $$\|\boldsymbol{x}\|_B = \text{minimum value } r \geq 0 \text{ such that } \boldsymbol{x} \in rB, \tag{1}$$
   where rB is just a scaling of the set B:
   $$\boldsymbol{x} \in B \quad \Rightarrow \quad r \cdot \boldsymbol{x} \in rB.$$

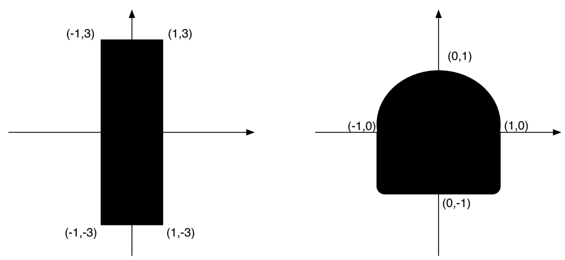
Last updated 10:51, September 2, 2016



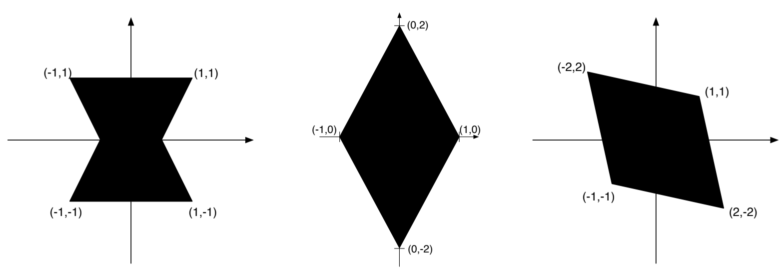
$B_1 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_1 \leq 1\}$ $B_2 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 \leq 1\}$ $B_\infty = \{\boldsymbol{x} : \|\boldsymbol{x}\|_\infty \leq 1\}$

(a) Let $\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. For $p = 1, 2, \infty$, find $r = \|\boldsymbol{x}\|_p$, and sketch $\boldsymbol{x}$ and rB_p (use different axes for each of the three values of p).

(b) Consider the 5 shapes below.



B_a B_b



B_c B_d B_e

Explain why $\|\cdot\|_{B_b}$ and $\|\cdot\|_{B_c}$ are **not** valid norms. The most convincing way to do this is to find vectors for which one of the three properties of a valid norm are violated.

(c) Give a concrete method for computing $\|\boldsymbol{x}\|_{B_a}, \|\boldsymbol{x}\|_{B_d}$, and $\|\boldsymbol{x}\|_{B_e}$ for any given vector $\boldsymbol{x}$. (For example: for B_1, which corresponds to the ℓ_1 norm, we would write $\|\boldsymbol{x}\|_1 = |x_1| + |x_2|$.) Using you expressions, show that these are indeed valid norms. This will require a little bit of thought.

5. Below, $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^N$.

(a) Prove that $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\|_\infty \cdot \|\boldsymbol{y}\|_1$.

(b) Prove that $\|\boldsymbol{x}\|_1 \leq \sqrt{N} \cdot \|\boldsymbol{x}\|_2$. (Hint: Cauchy-Schwarz)

(c) Let B_2 be the unit ball for the ℓ_2 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_2} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

Describe the vector x which achieves the maximum. (Hint: Cauchy-Schwarz)

(d) Let B_∞ be the unit ball for the ℓ_∞ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_\infty} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

(e) Let B_1 be the unit ball for the ℓ_1 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_1} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

(These last two might require some thought. If you solve them for $N = 2$, it should be easy to generalize.)

6. Let $\boldsymbol{A}$ be the 2×2 matrix

$$\boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 4 & 4 \\ -1/2 & 1/2 \end{bmatrix}.$$

For $\boldsymbol{x} \in \mathbb{R}^2$, define $\|\boldsymbol{x}\|_A = \|\boldsymbol{A}\boldsymbol{x}\|_2$.

(a) Show that $\|\cdot\|_A$ is indeed a valid norm.

(b) Sketch the unit ball $B_A = \{\boldsymbol{x} : \|\boldsymbol{x}\|_A \leq 1\}$ corresponding to $\|\cdot\|_A$. Feel free to use MATLAB.