ECE 3077, Summer 2014

Homework #6

Due Thursday July 3, in class

Reading: B&T 3.5

1. Using you class notes, prepare a 1–2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Let X be a random variable with pdf

$$f_X(x) = \begin{cases} x/4 & 1 \leq x \leq 3 \\ 0 & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 2\}$.

   (a) Find $E[X]$, $P(A)$, $f_{X|A}(x)$, and $E[X|A]$.

   (b) Let $Y = X^2$. Find $E[Y]$ and $\text{var}(Y)$.

3. You work for a certain three-letter government agency in Maryland and are monitoring telephone calls being placed through a particular cell tower. Let X denote the arrival time of the first call, as measured by the number of seconds after you arrive at work at 8am. Let Y denote the arrival time of the second call. In the most common model for X and Y used in the telephone industry, X and Y are continuous random variables with joint pdf

$$f_{X,Y}(x, y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \leq x < y \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant. Find the marginal pdfs $f_X(x)$ and $f_Y(y)$ and the conditional pdfs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

4. Suppose that X is uniform on $[0, 1]$. After obtaining a value of X, we generate $Y|X = x$ by picking a number uniformly at random from the interval $[x, 1]$. What is the marginal distribution of Y (i.e., $f_Y(y)$)?

5. Suppose that X is uniform on $[0, 1]$.

   (a) Calculate $\mathrm{E}\left[\frac{1}{X+1}\right]$.

   (b) Calculate $\mathrm{E}\left[\frac{X}{1-X}\right]$.

   (c) Let $Y = 1/(X + 1)$. Calculate $\mathrm{P}(Y \leq 3/4)$.

6. You are holding a stick of length ℓ. You choose a point uniformly at random along the stick and break it. You then take the **shorter** of the two pieces, choose a point along it uniformly at random and break it again.

   (a) Let L_1 be the length of the **shorter** piece after the first break. What are the cdf and pdf of L_1?

   (b) Let L_2 be the length of the **shortest** of all three pieces after the second break. What is the conditional pdf $f_{L_2|L_1}(l_2|l_1)$?

   (c) Calculate $\mathrm{E}[L_2]$.

7. A continuous random variable X has pdf

$$f_X(x) = e^{-x}, \; x \geq 0.$$

   We observe X in the presence of noise N:

$$Y = X + N.$$

   The noise is uniform; that is, given $X = x$, the conditional pdf of Y is uniform on $[x - 1/2, x + 1/2]$.

   (a) Find the marginal pdf $f_Y(y)$ of Y.

   (b) Suppose we observe $Y = 1/2$. What is the conditional pdf $f_X(x|Y = 1/2)$? Given this observation, what is the most likely value for X? (In other words, where does $f_X(x|Y = 1/2)$ take its maximum value?)

8. For each of the following pdfs, show whether X and Y are independent.

   (a)

$$f_{X,Y}(x, y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (b)

$$f_{X,Y}(x, y) = \begin{cases} 24xy & x \geq 0, y \geq 0, x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

9. Suppose that X_1 and X_2 are independent and identically distributed with pdf

$$f_X(x) = \begin{cases} 1 - x/2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

   Find

   (a) The joint pdf $f_{X_1,X_2}(x_1, x_2)$.

   (b) The cdf of $Z = \max(X_1, X_2)$.