Math. Program., Ser. B (2009) 116:499–512
DOI 10.1007/s10107-007-0115-7

FULL LENGTH PAPER

Augmented Lagrangians in semi-infinite programming

Jan-J. Rückmann · Alexander Shapiro

Received: 8 September 2005 / Accepted: 14 February 2006 / Published online: 3 May 2007

Abstract We consider the class of semi-infinite programming problems which became in recent years a powerful tool for the mathematical modeling of many real-life problems. In this paper, we study an augmented Lagrangian approach to semi-infinite problems and present necessary and sufficient conditions for the existence of corresponding augmented Lagrange multipliers. Furthermore, we discuss two particular cases for the augmenting function: the proximal Lagrangian and the sharp Lagrangian.

Keywords Semi-infinite programming · Augmented Lagrangian · Duality gap · Augmented Lagrange multiplier · Proximal Lagrangian · Sharp Lagrangian

Mathematics Subject Classification (2000) 90C34

1 Introduction

There are many applications from engineering and economics which can be modelled as a semi-infinite programming (SIP) problem, of the form

The work of this author was supported by CONACyT grant 44003.
The work of this author was partly supported by the NSF grant DMS-0510324.

J.-J. Rückmann
School of Mathematics, The University of Birmingham Edgbaston,
Birmingham B152TT, UK
e-mail: ruckmanj@maths.bham.ac.uk

A. Shapiro (✉)
School of Industrial and Systems Engineering, Georgia Institute of Technology,
Atlanta, GA 30332-0205, USA
e-mail: alex.shapiro@isye.gatech.edu

$$\operatorname*{Min}_{x\in\mathbb{R}^n} f(x) \text{ subject to } g(x,\omega)\le 0, \quad \omega\in\Omega, \tag{1.1}$$

where Ω is a (possibly infinite) nonempty set, $f:\mathbb{R}^n\to\mathbb{R}$ is a real valued function and $g:\mathbb{R}^n\times\Omega\to\mathbb{R}$. For a detailed discussion of the semi-infinite programming we refer to the recent monographs [1,3,13] as well as to the books [4,11] which contain tutorial papers on recent advances in semi-infinite programming.

In this paper we discuss an augmented Lagrangian approach to (possibly nonconvex) SIP problems of the form (1.1). For convex SIP problems there exists a well developed duality theory (e.g., [1,3,6]). The augmented Lagrangian approach with finitely many equality constraints was introduced by Hestenes [5] and Powell [7]. This was extended to finite dimensional inequality constrained problems by Buys [2]. Theoretical properties of the augmented Lagrangian duality method, in a finite dimensional setting with a finite number of constraints, were thoroughly investigated in Rockafellar [8,9], where, in particular, it was connected with the conjugate duality theory. In, a general setting, augmented Lagrangians are discussed in Rockafellar and Wets [10, Chap. 11, Sect. K*] (see also [14] for a recent survey of augmented Lagrangians in nonlinear programming).

This paper is organized as follows. Section 2 contains definitions of augmented Lagrange multipliers and augmented Lagrangians of the SIP problem (1.1), referred to as problem (P), and gives a general discussion of the corresponding duality properties. In Sect. 3 we investigate two particular cases of the augmenting function: the proximal Lagrangian and the sharp Lagrangian. In particular, we discuss existence of corresponding augmented Lagrange multipliers. Finally, Sect. 4 contains some illustrating examples.

Let us finally make the following remark. Of course, it is possible to replace the constraints $g(x,\omega)\le 0$, $\omega\in\Omega$, by one constraint $h(x)\le 0$, where $h(x) := \sup_{\omega\in\Omega} g(x,\omega)$. Under certain regularity conditions the max-function $h(x)$ can be represented as a maximum of a finite number of smooth functions and hence the SIP problem (1.1) can be reduced to a problem with a finite number of smooth constraints. This is a well known so-called “reduction approach". Regularity conditions, which are required for the reduction approach, are quite strong, e.g., the set Ω should be a sufficiently simple subset of a finite dimensional space, the function $g(x,\omega)$ should be smooth in x and ω, the set of active constraints should be finite, etc. The approach considered in this paper also involves a rather strong assumption of a finite discretization (see Theorem 1 below). The two approaches, the reduction and the one considered in this paper, are based on different ideas and require quite different regularity conditions, and hence could be applied to different situations.

2 Augmented Lagrange multipliers

We use the following framework throughout the paper. Consider a linear space $\mathcal{Y}$ of real valued functions $y:\Omega\to\mathbb{R}$. We assume that it satisfies the following condition.

(C) For any finite set $\{\omega_1,\dots,\omega_m\}\subset\Omega$ and any real numbers $\bar{y}_i$, $i=1,\dots,m$, there exists $y\in\mathcal{Y}$ such that $y(\omega_i)=\bar{y}_i$, $i=1,\dots,m$.

One possible example is to take $\mathcal{Y} := \mathbb{R}^{\Omega}$, i.e., the space of all functions $y : \Omega \to \mathbb{R}$. Another example, in case the set Ω is a metric space, is to take $\mathcal{Y} := C(\Omega)$, where $C(\Omega)$ denotes the space of all continuous functions $y : \Omega \to \mathbb{R}$. We assume throughout the paper that for every $x \in \mathbb{R}^n$ the function $g(x, \cdot)$ belongs to the space $\mathcal{Y}$.

Consider also the space of functions $\lambda : \Omega \to \mathbb{R}$ such that only a finite number of $\lambda(\omega)$ are nonzero. We denote this space by $\mathbb{R}^{(\Omega)}$, and for $\lambda \in \mathbb{R}^{(\Omega)}$ and $y \in \mathcal{Y}$ define the scalar product $\langle \lambda, y \rangle := \sum_{\omega \in \Omega} \lambda(\omega) y(\omega)$, where the summation is taken over such $\omega \in \Omega$ that $\lambda(\omega) \neq 0$. For a finite set $\sigma = \{\omega_1, \ldots, \omega_m\} \subset \Omega$ and $y \in \mathcal{Y}$ we denote by y^σ the restriction of function $y(\cdot)$ to σ, i.e., $y^\sigma(\omega) = y(\omega)$ if $\omega \in \sigma$, and $y^\sigma(\omega) = 0$ if $\omega \notin \sigma$. By $\text{supp}(\lambda)$ we denote the set of $\omega \in \Omega$ such that $\lambda(\omega) \neq 0$. Of course, we have that $\langle \lambda, y \rangle = \langle \lambda, y^\sigma \rangle$, where $\sigma := \text{supp}(\lambda)$.

We refer to problem (1.1) as problem (P) and assume throughout the paper that its optimal value, denoted $\text{val}(P)$, is *finite*. We associate with problem (P) the following problem, denoted (P_y),

$$\operatorname*{Min}_{x \in \mathbb{R}^n} f(x) \text{ subject to } g(x, \omega) + y(\omega) \leq 0, \quad \omega \in \Omega, \tag{2.1}$$

parameterized by $y \in \mathcal{Y}$. Denote by $v(y)$ the optimal value of problem (P_y), i.e., $v(y) := \text{val}(P_y)$. Consider also a nonnegative valued convex function $\alpha : \mathbb{R}^{(\Omega)} \to \mathbb{R}_+$ such that $\alpha(y) = 0$ iff $y = 0$. For $\lambda \in \mathbb{R}^{(\Omega)}$, $\sigma := \text{supp}(\lambda)$ and a real number $\tau \geq 0$ consider the function

$$v_\tau^\lambda(y) := v(y) + \tau \alpha(y^\sigma).$$

Note that function v_τ^λ is associated with a finite set $\sigma \subset \Omega$, which is supposed to be the support set of a considered element $\lambda \in \mathbb{R}^{(\Omega)}$. Also, with some abuse of notation, for an element $y \in \mathbb{R}^{(\Omega)}$ we use the same symbol y to denote a corresponding finite dimensional vector formed from the nonzero elements of y. Since an order in which components of $y \in \mathbb{R}^{(\Omega)}$ are used to form the corresponding finite dimensional vector is arbitrary, we assume that the function $\alpha(y)$ is invariant with respect to permutations of elements of y. We refer to the function $\alpha(y)$ as the *augmenting function*.

Definition 1 We say that $\lambda \in \mathbb{R}^{(\Omega)}$ is an augmented Lagrange multiplier of (P) if there exists $\tau \geq 0$ such that

$$v_\tau^\lambda(y) \geq v_\tau^\lambda(0) + \langle \lambda, y \rangle, \quad \forall\, y \in \mathcal{Y}. \tag{2.2}$$

The above definition is a natural extension of the Augmented Lagrangian approach discussed in Rockafellar and Wets [10, Chap. 11, Sect. K*]. Note that $v_\tau^\lambda(0) = v(0) = \text{val}(P)$. Interesting examples of the augmenting function are $\alpha(y) := \|y\|_2^2$ and $\alpha(y) := \|y\|_p$, where $y \in \mathbb{R}^{(\Omega)}$ and $\|y\|_p := \left(\sum_{\omega \in \text{supp}(y)} |y(\omega)|^p\right)^{1/p}$, $p \geq 1$. We discuss these two particular examples in more details in Sect. 3.

For $\lambda \in \mathbb{R}^{(\Omega)}$, $\sigma := \text{supp}(\lambda)$ and $\tau \geq 0$ we refer to

$$\mathcal{L}(x, \lambda, \tau) := \inf_{y \in \mathcal{Y}} \left\{ f(x) - \langle \lambda, y \rangle + \tau \alpha(y^\sigma) : g(x, \omega) + y(\omega) \leq 0,\ \omega \in \Omega \right\} \tag{2.3}$$

as the *augmented Lagrangian* of problem (P), associated with the augmenting function $\alpha(\cdot)$ (cf., [10, Definition 11.55]). Note that for $\tau = 0$ we have that $\mathcal{L}(x, \lambda, 0) = L(x, \lambda)$, if $\lambda \geq 0$, and $\mathcal{L}(x, \lambda, 0) = -\infty$ otherwise, where

$$L(x, \lambda) := f(x) + \sum_{\omega \in \Omega} \lambda(\omega) g(x, \omega)$$

is the (standard) Lagrangian of problem (P).

For $\lambda \in \mathbb{R}^{(\Omega)}$ and $\sigma := \text{supp}(\lambda)$ we have the following

$$\begin{aligned}\inf_{y \in \mathcal{Y}} \left\{ v_\tau^\lambda(y) - \langle \lambda, y \rangle \right\} &= \inf_{y \in \mathcal{Y}} \inf_{x \in \mathbb{R}^n} \big\{ f(x) - \langle \lambda, y \rangle + \tau \alpha(y^\sigma) : g(x, \omega) \\ &\qquad + y(\omega) \leq 0, \ \omega \in \Omega \big\} \\ &= \inf_{x \in \mathbb{R}^n} \inf_{y \in \mathcal{Y}} \big\{ f(x) - \langle \lambda, y \rangle + \tau \alpha(y^\sigma) : g(x, \omega) \\ &\qquad + y(\omega) \leq 0, \ \omega \in \Omega \big\},\end{aligned}$$

and hence

$$\inf_{y \in \mathcal{Y}} \left\{ v_\tau^\lambda(y) - \langle \lambda, y \rangle \right\} = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \tau).$$

We also have (cf. [10, Theorem 11.59]) that for any $\tau \geq 0$,

$$\sup_{\lambda \in \mathbb{R}^{(\Omega)}} \mathcal{L}(x, \lambda, \tau) = \begin{cases} f(x), & \text{if } g(x, \omega) \leq 0, \ \omega \in \Omega, \\ +\infty, & \text{otherwise}, \end{cases}$$

and hence

$$\text{val}(P) = \inf_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^{(\Omega)}} \mathcal{L}(x, \lambda, \tau). \tag{2.4}$$

For $\tau \geq 0$ consider the following dual, denoted (D_τ), of problem (P):

$$\operatorname*{Max}_{\lambda \in \mathbb{R}^{(\Omega)}} \left\{ \phi(\lambda, \tau) := \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \tau) \right\}. \tag{2.5}$$

By (2.4) it follows that $\text{val}(P) \geq \text{val}(D_\tau)$. Moreover, if there exists augmented Lagrange multiplier $\bar{\lambda}$, then (for respective $\tau \geq 0$) we have that (cf. [10, Theorem 11.59]): (i) $\text{val}(P) = \text{val}(D_\tau)$, (ii) $\bar{\lambda}$ is an optimal solution of (D_τ), (iii) if x_0 is a (globally) optimal solution of (P), then $(x_0, \bar{\lambda})$ is a saddle point of $\mathcal{L}(\cdot, \cdot, \tau)$ and $\text{val}(P) = \mathcal{L}(x_0, \bar{\lambda}, \tau)$. Conversely, if for some $\tau \geq 0$, $(x_0, \bar{\lambda})$ is a saddle point of $\mathcal{L}(\cdot, \cdot, \tau)$, then x_0 is an optimal solution of (P) and $\bar{\lambda}$ is an augmented Lagrange multiplier.

Now for a finite set $\sigma = \{\omega_1, \ldots, \omega_m\} \subset \Omega$ consider the corresponding discretization, denoted (P^σ), of problem (P):

$$\operatorname*{Min}_{x \in \mathbb{R}^n} f(x) \ \text{ subject to } \ g(x, \omega_i) \leq 0, \quad i = 1, \ldots, m.$$

With the above problem (P^σ) is associated the optimal value function $\hat{v}^\sigma(\bar{y})$, where $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m) \in \mathbb{R}^m$, given by the optimal value of the problem

$$\operatorname*{Min}_{x\in\mathbb{R}^n} f(x) \text{ subject to } g(x,\omega_i) + \bar{y}_i \le 0,\ i = 1,\dots,m. \tag{2.6}$$

We also consider the associated function $\hat{v}^\sigma_\tau(\bar{y}) := \hat{v}^\sigma(\bar{y}) + \tau\alpha(\bar{y})$, the corresponding augmented Lagrangian

$$\mathcal{L}^\sigma(x,\lambda,\tau) := \inf_{\bar{y}\in\mathbb{R}^m} \left\{ f(x) - \langle\lambda,\bar{y}\rangle + \tau\alpha(\bar{y}) : g(x,\omega_i) + \bar{y}_i \le 0,\ i = 1,\dots,m \right\},$$

$(x,\lambda,\tau) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$, and dual problem (D^σ_τ). Note that problem (2.6) is a relaxation of problem (2.1), provided that $y(\omega_i) = \bar{y}_i$, $i = 1,\dots,m$, and hence $v(y) \ge \hat{v}^\sigma(\bar{y})$ and consequently $v^\lambda_\tau(y) \ge \hat{v}^\sigma_\tau(\bar{y})$ for any $\lambda \in \mathbb{R}^{(\Omega)}$ such that $\operatorname{supp}(\lambda) = \sigma$.

Theorem 1 *Suppose that problem (P) has an augmented Lagrange multiplier $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$, and let $\sigma := \operatorname{supp}(\bar{\lambda})$. Then $\operatorname{val}(P) = \operatorname{val}(P^\sigma)$ and problems (P) and (P^σ) have the same set of optimal solutions.*

Proof Let us observe that for any $\tau \ge 0$, $\lambda \in \mathbb{R}^{(\Omega)}$ and $\sigma' := \operatorname{supp}(\lambda)$ we have that $\mathcal{L}^{\sigma'}(\cdot,\lambda,\tau) = \mathcal{L}(\cdot,\lambda,\tau)$. Because of the condition (C) this is a straightforward consequence of the definition of the augmented Lagrangians. Note also that for any finite set $\sigma' \subset \Omega$ we have that $\operatorname{val}(D^{\sigma'}_\tau) \le \operatorname{val}(D_\tau)$. Since $\bar{\lambda}$ is an optimal solution of (D_τ) for an appropriate $\tau \ge 0$, we obtain that $\bar{\lambda}$ is also an optimal solution of (D^σ_τ) and $\operatorname{val}(D^\sigma_\tau) = \operatorname{val}(D_\tau)$. We have then that

$$\operatorname{val}(P) \ge \operatorname{val}(P^\sigma) \ge \operatorname{val}(D^\sigma_\tau) = \operatorname{val}(D_\tau).$$

Since existence of the augmented Lagrange multiplier $\bar{\lambda}$ implies that $\operatorname{val}(P) = \operatorname{val}(D_\tau)$, we obtain that $\operatorname{val}(P) = \operatorname{val}(P^\sigma)$.

We have here that if x_0 is an optimal solution of problem (P), then $(x_0,\bar{\lambda})$ is a saddle point of $\mathcal{L}(\cdot,\cdot,\tau)$. It follows that $(x_0,\bar{\lambda})$ is a saddle point of $\mathcal{L}^\sigma(\cdot,\cdot,\tau)$ as well, and hence x_0 is an optimal solution of (P^σ) (recall that, with some abuse of the notation, we denote by the same $\bar{\lambda}$ an element of $\mathbb{R}^{(\Omega)}$ and the corresponding finite dimensional vector). Conversely, if x_0 is an optimal solution of problem (P^σ), then $(x_0,\bar{\lambda})$ is a saddle point of $\mathcal{L}^\sigma(\cdot,\cdot,\tau)$. It follows that $(x_0,\bar{\lambda})$ is a saddle point of $\mathcal{L}(\cdot,\cdot,\tau)$, and hence x_0 is an optimal solution of (P). □

Consider the following conditions:

(A1) There exists a finite set $\sigma = \{\omega_1,\dots,\omega_m\} \subset \Omega$ such that $\operatorname{val}(P) = \operatorname{val}(P^\sigma)$.
(A2) Problem (P^σ) possesses augmented Lagrange multiplier $\bar{\lambda}$.

By Theorem 2.1, we have that condition (A1) is necessary for existence of an augmented Lagrange multiplier for problem (P).

Note that $\hat{v}^\sigma_\tau(0) = \operatorname{val}(P^\sigma)$, and hence under assumption (A1) we have that $\hat{v}^\sigma_\tau(0) = v^\lambda_\tau(0)$, for any $\lambda \in \mathbb{R}^{(\Omega)}$. For $y \in \mathcal{Y}$ let $\bar{y}_i := y(\omega_i)$, $\omega_i \in \sigma$, and let $\lambda^* \in \mathbb{R}^{(\Omega)}$ be such that $\mathrm{supp}(\lambda^*) = \sigma$ and $\lambda^*(\omega_i) = \bar{\lambda}_i$, $i = 1, \ldots, m$. Then, under assumption (A2), we can write

$$v_\tau^{\lambda^*}(y) \geq \hat{v}_\tau^\sigma(\bar{y}) \geq \hat{v}_\tau^\sigma(0) + \sum_{i=1}^m \bar{\lambda}_i \bar{y}_i = v_\tau^{\lambda^*}(0) + \langle \lambda^*, y \rangle,$$

and hence λ^* is an augmented Lagrange multiplier of problem (P). Therefore, in order to establish existence of an augmented Lagrange multiplier for the semi-infinite programming problem (P), it suffices to verify conditions (A1) and (A2). Condition (A2) involves problem (P^σ) which is subject to a *finite* number of constraints.

3 Proximal and sharp Lagrangians

In this section we discuss augmented Lagrangians associated with the quadratic augmenting function $\alpha(y) := \sum_{\omega \in \Omega} y(\omega)^2 = \|y\|_2^2$, $y \in \mathbb{R}^{(\Omega)}$, referred to as *proximal* Lagrangian, and augmenting functions of the form $\alpha(y) := \|y\|$, where $\|\cdot\|$ is a norm on the corresponding finite dimensional space, referred to as *sharp* Lagrangians. We assume that a considered norm $\|y\|$ is invariant with respect to permutations of components of y, for example, one can use the ℓ_p-norms, $p \in [1, +\infty]$. In particular, we discuss existence of the corresponding augmented Lagrange multipliers.

3.1 Proximal Lagrangians

Unless stated otherwise we assume in this section that $\alpha(y) := \|y\|_2^2$. By using condition (C), it is straightforward to calculate (cf.[9]) that for $\tau > 0$,

$$\mathcal{L}(x, \lambda, \tau) = f(x) + \tau \sum_{\omega \in \mathrm{supp}(\lambda)} \psi(g(x, \omega), \lambda/\tau),$$

where

$$\begin{aligned}\psi(a, b) &:= \inf_{z \in \mathbb{R}} \left\{ -bz + z^2 : z \leq -a \right\} \\ &= \begin{cases} -b^2/4, & \text{if } b/2 \leq -a, \\ ba + a^2, & \text{if } b/2 > -a. \end{cases}\end{aligned}$$

Sufficient (and almost necessary) conditions for existence of augmented Lagrange multipliers for the reduced problem (P^σ) are known. In order to formulate such conditions consider the Lagrangian

$$L^\sigma(x, \lambda) := f(x) + \sum_{i=1}^m \lambda_i g(x, \omega_i), \quad x \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^m,$$

associated with problem (P^σ). Assume now that the functions $f(\cdot)$ and $g(\cdot,\omega), \omega \in \Omega$, are twice continuously differentiable. We use the following condition, due to Rockafellar [9], associated with problem (P^σ).

Condition (R^σ) There exist real constants a and b such that

$$\hat{v}^\sigma(\bar{y}) \geq a - b\|\bar{y}\|_2^2, \quad \forall\, \bar{y} \in \mathbb{R}^m. \tag{3.1}$$

If, for example, $f(x)$ is bounded on $\mathbb{R}^n$ from below by a constant c, then (3.1) holds with $a := c$ and $b := 0$. Note also that since in finite dimensional spaces any two norms are equivalent, it does not matter what norm is used in (3.1).

Let x_0 be a locally optimal solution of (P^σ) and $\bar{\lambda} \in \mathbb{R}^m$ be a corresponding vector of Lagrange multipliers. Denote by (P_y^σ) the corresponding problem perturbed by vector $y \in \mathbb{R}^m$, i.e., problem of the form (2.6). Then the following second-order conditions hold (cf. [9,12], see also [1, Chap. 3] for a general discussion of second-order optimality conditions).

(i) If $\bar{\lambda}$ is an augmented Lagrange multiplier of problem (P^σ), then the following second order condition holds:

$$D_{xx}^2 L^\sigma(x_0, \bar{\lambda})(h,h) \geq 0, \quad \forall\, h \in C(x_0).$$

Note that the above condition is a necessary condition for x_0 to be a locally optimal solution of (P^σ) if the Lagrange multipliers vector $\bar{\lambda}$ is unique (by $C(x_0)$ we denote the critical cone of problem (P^σ) at x_0).

(ii) Conversely, suppose that x_0 is a unique (globally) optimal solution of (P^σ), and that: (a) condition (R^σ) holds, (b) for all $y \in \mathbb{R}^n$ in a neighborhood of 0 problem (P_y^σ) has an optimal solution $\bar{x}(y)$ converging to x_0 as $y \to 0$, and (c) the following second order condition holds

$$D_{xx}^2 L^\sigma(x_0, \bar{\lambda})(h,h) > 0, \quad \forall\, h \in C(x_0) \setminus \{0\}. \tag{3.2}$$

Then $\bar{\lambda}$ is an augmented Lagrange multiplier of problem (P^σ).

Let us discuss now assumption (A1). First order necessary conditions for the semi-infinite problem (1.1) are well known (e.g., [1, Sect. 5.4.2]). That is, if x_0 is a locally optimal solution of the semi-infinite problem (P), then under some regularity conditions there exists $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$ such that

$$D_x L(x_0, \bar{\lambda}) = 0, \quad \bar{\lambda}(\omega) g(x_0, \omega) = 0 \text{ and } \bar{\lambda}(\omega) \geq 0, \quad \forall\, \omega \in \Omega, \tag{3.3}$$

and, moreover, $|\mathrm{supp}(\bar{\lambda})| \leq n$. Suppose that Ω is a nonempty compact metric space, functions $f(\cdot)$ and $g(\cdot,\omega), \omega \in \Omega$, are twice differentiable and $\nabla_{xx}^2 g(x,\omega)$ is continuous on $\mathbb{R}^n \times \Omega$. Then under the extended Mangasarian–Fromovitz constraint qualification, there exists a nonempty (and bounded) set of Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$ satisfying the first order optimality conditions (3.3).

Moreover, let $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$ be such that condition (3.3) holds for some $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$ and let $\sigma = \{\omega_1, \ldots, \omega_m\} := \mathrm{supp}(\bar{\lambda})$. Note that $L(\cdot, \bar{\lambda}) = L^\sigma(\cdot, \bar{\lambda})$, where we use the same notation $\bar{\lambda}$ for the m-dimensional vector $(\bar{\lambda}(\omega_1), \dots, \bar{\lambda}(\omega_m))$, and hence x_0 satisfies the first order (KKT) optimality conditions for the corresponding problem (P^σ) as well. Suppose, further, that the second order sufficient condition (3.2) holds. Then x_0 is a locally optimal solution of the problem (P^σ). Note that condition (3.2) does not take into account the so-called sigma term and therefore is stronger than the corresponding second order sufficient conditions for the semi-infinite problem (P) (see [1, Sect. 5.4.3] for a discussion of second order optimality conditions of SIP problems). Of course, in order to ensure that x_0 is a globally optimal solution of the reduced problem (P^σ) we would need some additional assumptions.

Theorem 2 *Suppose that the semi-infinite problem* (P) *possesses unique (globally) optimal solution* x_0 *and there exists* $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$ *satisfying the first order necessary condition* (3.3)*, and let* $\sigma := \mathrm{supp}(\bar{\lambda})$. *Then the following holds.*

(a) *If* $\bar{\lambda}$ *is an augmented Lagrange multiplier of* (P)*, then* $\mathrm{val}(P) = \mathrm{val}(P^\sigma)$ *and* x_0 *is the unique globally optimal solution of* (P^σ).

(b) *Conversely, if* x_0 *is a unique (globally) optimal solution of* (P^σ)*, condition* (R^σ) *holds, for all* $y \in \mathbb{R}^n$ *in a neighborhood of* $0 \in \mathbb{R}^n$ *problem* (P^σ_y) *has an optimal solution* $\bar{x}(y)$ *converging to* x_0 *as* $y \to 0$*, and the second order sufficient condition* (3.2) *holds, then* $\bar{\lambda}$ *is an augmented Lagrange multiplier of problem* (P).

Proof The assertion (a) follows directly from Theorem 1.

Now if x_0 is a (globally) optimal solution of (P^σ), then $\mathrm{val}(P) = \mathrm{val}(P^\sigma)$. We also have here (see the above property (ii)) that under the assumptions of assertion (b), $\bar{\lambda}$ is an augmented Lagrange multiplier of problem (P^σ). It follows that $\bar{\lambda}$ is an augmented Lagrange multiplier of problem (P) as well. □

Remark 1 It is possible to give a local version of the above theorem. Assume again that Ω is a compact metric space and $\mathcal{Y} := C(\Omega)$ is the space of continuous functions $y : \Omega \to \mathbb{R}$ equipped with the sup-norm $\|y\|_\infty := \sup_{\omega\in\Omega} |y(\omega)|$. Suppose that the semi-infinite problem (P) possesses unique (globally) optimal solution x_0 and there exists $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$ satisfying the first order necessary condition (3.3), and let $\sigma := \mathrm{supp}(\bar{\lambda})$. Suppose, further, that for all $y \in \mathcal{Y}$ in a neighborhood of $0 \in \mathcal{Y}$ problem (P_y) has an optimal solution $\bar{x}(y)$ converging to x_0 as $\|y\|_\infty \to 0$, and the second order sufficient condition (3.2) holds. By the second order condition (3.2) we have that x_0 is a locally optimal solution of (P^σ), and hence by the above discussion that there exist constants $\gamma > 0$ and $\tau \geq 0$ such that the inequality (2.2) holds for all $y \in \mathcal{Y}$ satisfying $\|y\|_\infty \leq \gamma$. Let us replace the augmented Lagrangian $\mathcal{L}(x, \lambda, \tau)$, defined in (2.3), with the following (locally restricted) augmented Lagrangian

$$\mathcal{L}(x, \lambda, \tau, \gamma) := \inf_{y\in\mathcal{Y},\ \|y\|_\infty\leq\gamma} \left\{ f(x) - \langle\lambda, y\rangle + \tau\alpha(y^\sigma) : g(x, \omega) + y(\omega) \leq 0,\ \omega\in\Omega \right\}.$$

We can write then the corresponding dual problem $D_{\tau,\gamma}$ defined in the same way as in (2.5) only with $\mathcal{L}(x, \lambda, \tau, \gamma)$ instead of $\mathcal{L}(x, \lambda, \tau)$. It follows that $\mathrm{val}(P) = \mathrm{val}(D_{\tau,\gamma})$ and $\bar{\lambda}$ is an optimal solution of $D_{\tau,\gamma}$. Let us finally note that the assumption that "$\bar{x}(y)$ converges to x_0 as $y \to 0$" holds automatically if x_0 is unique optimal solution of

(P) and optimization of (P_y) can be restricted to a compact subset of $\mathbb{R}^n$ for all y in a neighborhood of $0 \in \mathcal{Y}$ (this is the so-called inf-compactness condition).

3.2 Sharp Lagrangians

Assume now that the augmenting function is of the form $\alpha(\cdot) := \|\cdot\|$. Denote by $\|\cdot\|^*$ the dual norm of the above norm. Note that for a vector $y \in \mathcal{Y}$ we have that $\|y^\sigma\| = \sup_{\|z^\sigma\|^* \le 1} \langle z^\sigma, y\rangle$. Then for $\lambda \in \mathbb{R}^{(\Omega)}$, $\sigma := \mathrm{supp}(\lambda) = \{\omega_1, \dots, \omega_m\}$ and $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
&\inf_{y\in\mathcal{Y}} \Big\{ -\langle\lambda, y\rangle + \tau\|y^\sigma\| : g(x,\omega) + y(\omega) \le 0,\ \omega \in \sigma \Big\} \\
&\quad = \inf_{y\in\mathcal{Y}} \sup_{\|z^\sigma\|^*\le 1} \Big\{ -\langle\lambda, y\rangle + \tau\langle z^\sigma, y\rangle : g(x,\omega) + y(\omega) \le 0,\ \omega \in \sigma \Big\} \\
&\quad = \sup_{\|z^\sigma\|^*\le 1} \inf_{y\in\mathcal{Y}} \Big\{ -\langle\lambda, y\rangle + \tau\langle z^\sigma, y\rangle : g(x,\omega) + y(\omega) \le 0,\ \omega \in \sigma \Big\}.
\end{aligned} \tag{3.4}$$

The interchange of "inf" and "sup" operators in the above can be justified, for example, by applying [15, Theorem 2.10.2]. Furthermore, by making change of variables $u_i = -[g(x,\omega_i) + y(\omega_i)]$, $i = 1, \dots, m$, and denoting $u = (u_1, \dots, u_m)$ and $g^\sigma(x) := (g(x,\omega_1), \dots, g(x,\omega_m))$, we obtain

$$\begin{aligned}
&\inf_{y\in\mathcal{Y}} \Big\{ -\langle\lambda, y\rangle + \tau\|y^\sigma\| : g(x,\omega) + y(\omega) \le 0,\ \omega \in \sigma \Big\} \\
&\quad = \sup_{\|z^\sigma\|^*\le 1} \inf_{u\ge 0} \Big\{ \langle\lambda, u + g^\sigma(x)\rangle - \tau\langle z^\sigma, u + g^\sigma(x)\rangle \Big\} \\
&\quad = \langle\lambda, g^\sigma(x)\rangle - \tau \inf_{\substack{\|z^\sigma\|^*\le 1 \\ \lambda - \tau z^\sigma \ge 0}} \langle z^\sigma, g^\sigma(x)\rangle \\
&\quad = \langle\lambda, g^\sigma(x)\rangle + \tau \sup_{\substack{\|z^\sigma\|^*\le 1 \\ \lambda + \tau z^\sigma \ge 0}} \langle z^\sigma, g^\sigma(x)\rangle.
\end{aligned} \tag{3.5}$$

We obtain that

$$\mathcal{L}^\sigma(x,\lambda,\tau) = L^\sigma(x,\lambda) + \tau \sup_{z\in\mathbb{R}^{(\Omega)}} \Big\{ \langle z, g(x,\omega)\rangle : \|z\|^* \le 1,\ \tau z + \lambda \ge 0,\ \mathrm{supp}(z) \subset \sigma \Big\}. \tag{3.6}$$

In particular, if the chosen norm is ℓ_1-norm $\|\cdot\|_1$, and hence its dual $\|\cdot\|_1^*$ is the max-norm $\|\cdot\|_\infty$, then for $\lambda = 0$ we have

$$\mathcal{L}^\sigma(x,0,\tau) = f(x) + \tau \max\Big\{ [g(x,\omega)]_+ : \omega \in \sigma \Big\}, \tag{3.7}$$

where $[a]_+ := \max\{0, a\}$. The second term in the right hand side of (3.7) can be viewed as a penalty term.

As far as existence of augmented Lagrange multipliers, for the reduced problem (P^σ), is concerned we have the following result. Suppose that: (B1) $f(\cdot)$ and $g(\cdot,\omega)$, $\omega \in \Omega$, are continuously differentiable, (B2) the inf-compactness condition, for the reduced problem, holds. Then the reduced problem has a nonempty compact set of optimal solutions, denoted by $\mathcal{S}$. Suppose, further, that: (B3) the Mangasarian–Fromovitz Constraint Qualification (MFCQ), for the reduced problem, holds at every $x \in \mathcal{S}$. Then to every point $x \in \mathcal{S}$ corresponds a nonempty and compact set of (standard) Lagrange multipliers, denoted by $\Lambda^\sigma(x)$, of the reduced problem. Note that, under assumptions (B1)–(B3), the set $\cup_{x\in\mathcal{S}}\Lambda^\sigma(x)$ is bounded. This follows, by compactness of $\mathcal{S}$, from the fact that the MFCQ implies that the sets $\Lambda^\sigma(x)$ are uniformly bounded for x in a neighborhood of any point $\bar{x} \in \mathcal{S}$ (e.g., [1, Proposition 4.43]). Also the following result holds (cf. [1, Theorem 4.26]):

$$\hat{v}^\sigma(\bar{y}) \geq \hat{v}^\sigma(0) + \inf_{x\in\mathcal{S}} \inf_{\lambda\in\Lambda^\sigma(x)} \langle\lambda, \bar{y}\rangle + o(\|\bar{y}\|).$$

Moreover, suppose that: (B4) $\Lambda^\sigma(x) = \{\bar{\lambda}(x)\}$ is a singleton, for every $x \in \mathcal{S}$. Then

$$\hat{v}^\sigma(\bar{y}) = \hat{v}^\sigma(0) + \inf_{x\in\mathcal{S}} \langle\bar{\lambda}(x), \bar{y}\rangle + o(\|\bar{y}\|).$$

We have that for any (fixed) $\lambda^* \in \mathbb{R}^m$,

$$\begin{aligned}\inf_{x\in\mathcal{S}} \inf_{\lambda\in\Lambda^\sigma(x)} \langle\lambda, \bar{y}\rangle = \langle\lambda^*, \bar{y}\rangle - \sup_{x\in\mathcal{S},\, \lambda\in\Lambda^\sigma(x)} \langle\lambda^* - \lambda, \bar{y}\rangle \geq \langle\lambda^*, \bar{y}\rangle \\ - \sup_{x\in\mathcal{S},\, \lambda\in\Lambda^\sigma(x)} \|\lambda - \lambda^*\|^* \, \|\bar{y}\|.\end{aligned}$$

Assuming that the above conditions (B1)–(B3) hold, we obtain that for any $\lambda^* \in \mathbb{R}^m$ and

$$\tau > \sup_{\lambda\in\bigcup_{x\in\mathcal{S}}\Lambda^\sigma(x)} \|\lambda - \lambda^*\|^*, \tag{3.8}$$

(recall that the set $\cup_{x\in\mathcal{S}}\Lambda^\sigma(x)$ is bounded here and hence the right hand side of (3.8) is finite) there exists a neighborhood N_τ of $0 \in \mathbb{R}^m$ such that

$$\hat{v}^\sigma_\tau(\bar{y}) \geq \hat{v}^\sigma(0) + \langle\lambda^*, \bar{y}\rangle, \quad \forall \bar{y} \in N_\tau.$$

The above inequality gives a local result (compare with the discussion of Remark 1). Under the above conditions (B1)–(B4), we obtain that

$$\tau \geq \sup_{x\in\mathcal{S}} \|\bar{\lambda}(x) - \lambda^*\|^*$$

is a necessary condition for λ^* to be an augmented Lagrange multiplier, of problem (P^σ), with the corresponding coefficient τ.

Theorem 3 *Suppose that the semi-infinite problem* (P) *possesses a locally optimal solution* x_0*, there exists* $\bar{\lambda} \in \mathbb{R}^{(\Omega)}$ *satisfying the first order necessary condition* (3.3)*, the second order sufficient condition* (3.2) *holds and the above conditions* (B1)–(B3) *are satisfied. Set* $\sigma := \operatorname{supp}(\bar{\lambda})$. *Then for any* $\lambda^* \in \mathbb{R}^m$ *and* τ *satisfying* (3.8) *it follows that* x_0 *is a locally optimal solution of the problem*

$$\operatorname*{Min}_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda^*, \tau).$$

In particular, by taking the ℓ_1*-norm* $\|\cdot\|_1$ *and* $\lambda^* = 0$ (*recall that* $\|\cdot\|_1^* = \|\cdot\|_\infty$), *we obtain that for*

$$\tau > \sup\left\{\|\lambda\|_\infty : \lambda \in \Lambda^\sigma(x),\ \ x \in \mathcal{S}\right\},$$

x_0 *is a locally optimal solution of the problem*

$$\operatorname*{Min}_{x \in \mathbb{R}^n} f(x) + \tau \max\left\{[g(x, \omega)]_+ : \omega \in \sigma\right\}. \tag{3.9}$$

Remark 2 We also can consider the following problem

$$\operatorname*{Min}_{x \in \mathbb{R}^n} f(x) + \tau \sup\left\{[g(x, \omega)]_+ : \omega \in \Omega\right\}, \tag{3.10}$$

with a penalty term stronger than in (3.9). In case Ω is a compact metric space and $\mathcal{Y} = C(\Omega)$, it is known that if x_0 is a locally optimal solution of the semi-infinite problem (P) and the extended Mangasarian–Fromovitz constraint qualification holds at x_0, then x_0 is a locally optimal solution of the penalized problem (3.9) for $\tau > 0$ large enough (cf. [1, Proposition 3.111]). Example 4.3 in the following section illustrates the difference between penalizations (3.9) and (3.10).

4 Examples

This section contains three illustrating examples. Examples 1 and 2 discuss existence of augmented Lagrange multipliers. By Theorem 2.1, existence of a finite discretization of problem (P) with the same optimal value (condition (A1)) is necessary for existence of an augmented Lagrange multiplier. Note that condition (A1) may not hold even for linear SIP (see, e.g., [1, Examples 5.102, 5.103]). Nevertheless, for convex SIP condition (A1) can be ensured by mild regularity conditions (see, e.g., [1, discussion on pages 502–506]). Let us consider the following examples of nonconvex SIP problems.

Example 1 Consider the following problem (P):

$$\operatorname{Min}\left\{f(x) := -x_1^2 - \tfrac{1}{4}x_2^2\right\} \text{ subject to } x_1\omega_1 + x_2\omega_2 - 1 \le 0,\ \ \omega \in \Omega, \tag{4.1}$$

with the index set $\Omega := \left\{\omega \in \mathbb{R}^2 : \omega_1^2 + \omega_2^2 = 1\right\}$. Here the index set Ω is a compact subset of $\mathbb{R}^2$ and we take $\mathcal{Y} := C(\Omega)$. We have that the maximum of $x_1\omega_1 + x_2\omega_2$, over $\omega \in \Omega$, is $\sqrt{x_1^2 + x_2^2}$, and hence the feasible set of problem (4.1) is

$\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}$. It is not difficult to verify that the point $x_0 = (1, 0)$ is a globally optimal solution of the above problem and $\mathrm{val}(P) = -1$. The only active constraint of problem (4.1) at the point x_0 is $x_1 \le 1$ (corresponding to the index $\omega_0 = (1, 0)$) with the associated Lagrange multiplier

$$\bar{\lambda}(\omega) = \begin{cases} 2, & \text{if } \omega = \omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

We have here that, for $\sigma := \{\omega_0\}$, the point x_0 is not a locally optimal solution of the corresponding finite problem (P^σ):

$$\text{Min } -x_1^2 - \tfrac{1}{4}x_2^2 \text{ subject to } x_1 \le 1, \tag{4.2}$$

and the condition (A1) is not fulfilled (otherwise x_0 would also be a locally optimal solution of (P^σ)). And, indeed, the above problem (P) does not possess augmented Lagrange multipliers. In order to verify this directly consider

$$y_\varepsilon(\omega) := -\max\{0, \omega_1 + \varepsilon\omega_2 - 1\}, \ \omega \in \Omega,$$

and the corresponding perturbed problem (P_{y_ε}). We have that for any $\varepsilon \in \mathbb{R}$ the point $x_\varepsilon := (1, \varepsilon)$ is a feasible solution of (P_{y_ε}), and $f(x_\varepsilon) = -1 - \frac{1}{4}\varepsilon^2$ and $y_\varepsilon^\sigma = 0$. It follows that for any $\lambda \in \mathbb{R}^{(\Omega)}$ supported on the set σ,

$$v_\tau^\lambda(y_\varepsilon) - \langle\lambda, y_\varepsilon\rangle = v(y_e) < -1 = v(0)$$

for any $\tau \ge 0$ and $\varepsilon \ne 0$. Consequently, any $\lambda \in \mathbb{R}^{(\Omega)}$ with $\mathrm{supp}(\lambda) = \sigma$, cannot be an augmented Lagrange multiplier of (P) (for any augmenting function). Note that such λ is not an augmented Lagrange multiplier even if the feasible set of problem (P) is restricted to a neighborhood of the point x_0.

On the other hand, the above problem (P) can be written as the following finite problem:

$$\text{Min } -x_1^2 - \tfrac{1}{4}x_2^2 \text{ subject to } x_1^2 + x_2^2 \le 1, \tag{4.3}$$

and the Lagrange multiplier corresponding to the solution $x_0 = (1, 0)$ is an augmented Lagrange multiplier of the finite problem (4.3).

In the following example the condition (R^σ) is not fulfilled and there does not exist an augmented Lagrange multiplier. However, if we restrict our consideration to an appropriate neighborhood of the feasible set, then (R^σ) becomes fulfilled and we also obtain an augmented Lagrange multiplier.

Example 2 Consider the following problem (P):

$$\text{Min}\left\{f(x) := x_1^3 + x_1^2 + x_2^2\right\} \text{ subject to } 2x_2\omega_2 - x_1 - \omega_1 + 2 \le 0, \ \omega \in \Omega, \tag{4.4}$$

with the index set $\Omega := \{\omega \in \mathbb{R}^2 : \omega_2^2 - \omega_1 + 1 = 0\}$, $\mathcal{Y} := \mathbb{R}^\Omega$ and quadratic augmenting function $\alpha(y) := \|y\|_2^2$. By calculating the maximum of the constraint function over $\omega \in \Omega$, it is possible to verify that the feasible set of the above problem (P) is

$\{x \in \mathbb{R}^2 : 2x_2^2 - x_1 + 1 \leq 0\}$. Then it is not difficult to see that the point $x_0 := (1, 0)$ is a globally optimal solution of problem (P) and $\text{val}(P) = 2$. The only active constraint at x_0 is $x_1 \geq 1$ (corresponding to the index $\omega_0 := (1, 0)$) with the associated Lagrange multiplier $\bar{\lambda}(\omega_0) = 5$ supported on $\sigma := \{\omega_0\}$. We have here that x_0 is also a (globally) optimal solution of the corresponding finite problem (P^σ) and $\text{val}(P) = \text{val}(P^\sigma)$, and hence the condition (A1) is fulfilled.

Now, consider the point $(s, 0)$, $s < 1$ (which is not feasible for (P)) and construct a corresponding perturbed problem (P_{y_s}) with

$$y_s(\omega) := -\max\{0, -s - \omega_1 + 2\}, \quad \omega \in \Omega.$$

Then, $(s, 0)$ is a feasible point of (P_{y_s}) with

$$v(y_s) = \begin{cases} 0, & \text{if } s \in [-1, 0], \\ s^3 + s^2, & \text{otherwise,} \end{cases}$$

and, for $s \leq -1$,

$$v_\tau(y_s) = s^3 + s^2 + \tau(s-1)^2$$

as well as

$$v_\tau(0) + \langle \bar{\lambda}, y_s \rangle = 2 + 5(s-1).$$

The latter two equalities imply that

$$\bar{\lambda}(\omega) = \begin{cases} 5, & \text{if } \omega = \omega_0, \\ 0, & \text{otherwise,} \end{cases}$$

is not an augmented Lagrange multiplier of (P). In particular, the condition (R^σ) is not fulfilled. However, if we restrict our consideration to a neighborhood of the feasible set of (P) which is a subset of the halfspace $\{x \in \mathbb{R}^2 : x_1 \geq s_0\}$ for an $s_0 \in \mathbb{R}$, then $f(x)$ becomes bounded from below and λ_0 becomes an augmented Lagrange multiplier.

The next example illustrates the difference between penalizations (3.9) and (3.10).

Example 3 Let $x \in \mathbb{R}^2$ and consider the following problem (P):

$$\text{Min}\left\{f(x) := -x_1^2 + x_2\right\} \text{ subject to } 2\omega x_1 - x_2 - \omega^2 \leq 0, \quad \omega \in \Omega := [-1, 1], \tag{4.5}$$

with $\mathcal{Y} := C(\Omega)$. We have

$$\sup_{\omega \in \Omega}\left\{2\omega x_1 - x_2 - \omega^2\right\} = \begin{cases} x_1^2 - x_2, & \text{if } |x_1| \leq 1, \\ 2|x_1| - x_2 - 1, & \text{otherwise.} \end{cases}$$

Therefore, in a sufficiently small neighborhood of the point $x_0 := (0, 0)$, the feasible region of problem (4.5) is defined by the constraint $x_1^2 - x_2 \leq 0$. Consequently, for feasible $x \in \mathbb{R}^2$ near x_0 we have that $f(x) \geq 0$, and hence x_0 is a locally optimal solution of (P). The first-order optimality conditions (3.3) hold at x_0 with unique Lagrange multiplier $\bar{\lambda}(0) = 1$, $\text{supp}(\bar{\lambda}) = \sigma$, with $\sigma := \{0\}$. However, it is not difficult to verify that x_0 is not a locally optimal solution of the reduced problem (P^σ) for any finite set $\sigma \subset \Omega$. It is also not difficult to see that x_0 is not a locally optimal solution of problem (3.9) for any $\tau > 0$. On the other hand, for $x \in \mathbb{R}^2$ near $(0, 0)$ we have

$$f(x) + \tau \sup \left\{ [g(x, \omega)]_+ : \omega \in [-1, 1] \right\} = -x_1^2 + x_2 + \tau \max\{0, x_1^2 - x_2\},$$

and x_0 is a locally optimal solution of the corresponding problem (3.10).

Acknowledgments The authors are indebted to anonymous referees for careful reading of the manuscript and many comments and suggestions which helped to improve the presentation.

References

1. Bonnans, J.F., Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer, Heidelberg (2000)
2. Buys, J.D.: Dual Algorithms for Constrained Optimization Problems. PhD Thesis, University of Leiden, Leiden (1972)
3. Goberna, M.A., López, M.A.: Linear Semi-Infinite Optimization. Wiley, Chichester (1998)
4. Goberna, M.A., López, M.A.: Semi-Infinite Programming—Recent Advances. Kluwer, Boston (2001)
5. Hestenes, M.R.: Multiplier and gradient methods. J. Optim. Theory Appl. **4**, 303–320 (1969)
6. Hettich, R., Kortanek, K.O.: Semi-infinite programming: theory, methods, and applications. SIAM Rev. **35**, 380–429 (1993)
7. Powell, M.J.D.: A method for nonlinear constraints in minimization problems. In: Fletcher, R., (ed.) Optimization. Academic, London (1969)
8. Rockafellar, R.T.: Conjugate Duality and Optimization. Regional Conference Series in Applied Mathematics. SIAM, Philadelphia (1974)
9. Rockafellar, R.T.: Augmented Lagrange multiplier functions and duality in convex programming. SIAM J. Control **12**, 268–285 (1974)
10. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer, Heidelberg (1998)
11. Reemtsen, R., Rückmann, J.-J. (eds.): Semi-Infinite Programming. Kluwer, Boston (1998)
12. Shapiro, A., Sun, J.: Some properties of the augmented Lagrangian in cone constrained optimization. Math. Oper. Res. **29**, 479–491 (2004)
13. Stein, O.: Bi-level Strategies in Semi-Infinite Programming. Kluwer, Boston (2003)
14. Sun, X.L., Li, D., Mckinnon, K.I.M.: On saddle points of augmented Lagrangians for constrained nonconvex optimization. SIAM J. Optim. **15**, 1128–1146 (2005)
15. Zalinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, New Jersey (2002)