Chi-bar-squared distributions

Recall that the largest linear space L contained in a convex closed cone $C \subset \mathbb{R}^p$, denoted $L = Lin(C)$, is called the *lineality space* of C.

Let V be a $p \times p$ positive definite symmetric matrix, $Y \sim N(0, V^{-1})$ be normally distributed random vector and $C_1, C_2 \subset \mathbb{R}^p$ be convex closed cones such that

$$C_1 \subset Lin(C_2). \tag{1}$$

Let us show that the statistic

$$T = \min_{\theta \in C_1} (Y - \theta)^T V (Y - \theta) - \min_{\theta \in C_2} (Y - \theta)^T V (Y - \theta) \tag{2}$$

has a $\bar{\chi}^2$ distribution.

Consider the scalar product $\langle x, y \rangle = x^T V y$ and the corresponding norm $\|x\| = \langle x, x \rangle^{1/2}$. Then the above statistic can be written as

$$T = \|P(Y, C_2)\|^2 - \|P(Y, C_1)\|^2,$$

where $P(Y, C)$ denotes the projection of Y onto the cone C, with respect to the considered scalar product. Let L be the lineality space of C_2 and $C^* = C_2 \cap L^\perp$, where $L^\perp$ denotes the orthogonal complement linear space of the space L with respect to the considered scalar product. We have then that $P(Y, C_2) = P(Y, L) + P(Y, C^*)$ and

$$\|P(Y, C_2)\|^2 = \|P(Y, C^*)\|^2 + \|P(Y, L)\|^2.$$

Moreover, since $C_1 \subset L$, we also have that

$$\|P(Y, L)\|^2 = \|P(Y, C_1)\|^2 + \|P(Y, C')\|^2,$$

where

$$C' = \{\theta \in L : \langle \theta, \eta \rangle \leq 0, \ \forall \, \eta \in C_1\}.$$

Consider the cone $C = C^* + C'$. Note that the cones C^* and C' lie in orthogonal spaces, and hence

$$\|P(Y, C)\|^2 = \|P(Y, C^*)\|^2 + \|P(Y, C')\|^2.$$

It follows that the above statistic can be written as

$$T = \|P(Y, C)\|^2,$$

and hence T has a $\bar{\chi}^2$ distribution.

If one of the cones C_1 or C_2 is a linear space, the above condition (1) simply means that $C_1 \subset C_2$.