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ON DIFFERENTIABILITY OF METRIC PROJECTIONS IN $\mathbf{R}^n$, 1: BOUNDARY CASE

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ABSTRACT. This paper is concerned with metric projections onto a closed subset S of a finite-dimensional normed space. Necessary and in a sense sufficient conditions for directional differentiability of a metric projection at a boundary point of S are given in terms of approximating cones. It is shown that if S is defined by a number of inequality constraints and a constraint qualification holds, then the approximating cone exists.

1. Introduction. Let $\|\cdot\|$ be a norm defined on the n-dimensional real vector space $\mathbf{R}^n$ and S be a closed subset of $\mathbf{R}^n$. The set-valued metric projection onto S is the point-to-set mapping $\Omega\colon \mathbf{R}^n \rightrightarrows S$ which corresponds to an $x \in \mathbf{R}^n$ the set of elements of S closest to x, that is

$$\Omega x = \{y \in S\colon \|x - y\| = d(x, S)\},$$

where $d(x, S)$ denotes the distance from x to S,

$$d(x, S) = \inf\{\|x - y\|\colon y \in S\}.$$

For every x the set Ωx is nonempty and compact, although possibly is not a singleton. We consider a selection mapping $P_S\colon \mathbf{R}^n \to S$ associated with Ω, which is defined as assigning to x a closest point $y = P_S x$ in S, so that $y \in \Omega x$. One may note that if $x \in S$, then $P_S x = x$ and P_S is continuous at x.

It is well known that if S is convex and the norm is strictly convex, then Ωx is a singleton and hence $P_S x$ is uniquely defined for all x. In this case it was shown in a number of publications that P_S is directionally differentiable at every $x \in S$, e.g., [**5**, **7**, **8**, **12**]. In this article we investigate P_S at a boundary point of S for an arbitrary closed set S. We give necessary and in a sense sufficient conditions for directional differentiability of P_S in terms of approximating cones. It will be shown that if S is defined by a number of inequality constraints, then under mild assumptions the approximating cone exists and thus P_S is directionally differentiable.

2. Cone approximation and directional differentiability of metric projections. The following concept of approximating cones has proved to be useful, e.g., in sensitivity analysis of nonlinear programs (see Shapiro [**11**]) and in deriving some asymptotic results in inequality constrained estimation [**2**, **10**].

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DEFINITION 1. *We say the the set S is approximated at $x_0 \in S$ by a closed cone $\mathcal{C}$, called an approximating cone, if*

$$\lim_{\substack{x \to x_0 \\ x \in S}} \frac{d(x - x_0, \mathcal{C})}{\|x - x_0\|} = 0 \tag{2.1}$$

and

$$\lim_{\substack{y \to 0 \\ y \in \mathcal{C}}} \frac{d(x_0 + y, S)}{\|y\|} = 0. \tag{2.2}$$

We recall that a function (or mapping) $f(x)$ is said to be directionally differentiable at a point x if the limit (directional derivative)

$$f'(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}$$

exists for all $y \in \mathbf{R}^n$. In the following theorem we give a characterization of approximating cones in terms of directional differentiability of the distance function.

THEOREM 1. *The approximating cone $\mathcal{C}$, to S at $x_0 \in S$, exists iff the distance function $\delta(\cdot) = d(\cdot, S)$ is directionally differentiable at x_0, in which case*

$$\delta'(x_0, \cdot) = d(\cdot, \mathcal{C}). \tag{2.3}$$

PROOF. First we observe that

$$\|P_S x - x_0\| \le \|P_S x - x\| + \|x - x_0\| \le 2\|x - x_0\| \tag{2.4}$$

and hence P_S is continuous at x_0. Now suppose that the approximating cone $\mathcal{C}$ exists. Then it follows from (2.1) that there is $y = y(x) \in \mathcal{C}$ such that

$$\|P_S x - x_0 - y\| = o(\|P_S x - x_0\|),$$

which by (2.4) becomes

$$\|P_S x - x_0 - y\| = o(\|x - x_0\|). \tag{2.5}$$

We have that

$$d(x, S) = \|x - P_S x\| \ge \|x - x_0 - y\| - \|P_S x - x_0 - y\|. \tag{2.6}$$

Noting that $d(x - x_0, \mathcal{C}) \le \|x - x_0 - y\|$ and applying (2.5) we obtain from (2.6) that

$$d(x, S) \ge d(x - x_0, \mathcal{C}) + o(\|x - x_0\|).$$

In a similar way it follows from (2.2) that

$$d(x - x_0, \mathcal{C}) \ge d(x, S) + o(\|x - x_0\|)$$

and hence

$$d(x, S) = d(x - x_0, \mathcal{C}) + o(\|x - x_0\|). \tag{2.7}$$

Since $\mathcal{C}$ is a cone, the function $d(\cdot, \mathcal{C})$ is positively homogeneous. Together with (2.7) this implied that for every y

$$d(x_0 + ty, S) = td(y, \mathcal{C}) + o(t),$$

i.e. $\delta(\cdot)$ is directionally differentiable at x_0 and (2.3) follows.

Conversely, suppose that $\delta(\cdot)$ is directionally differentiable at x_0 and consider the cone

$$\mathcal{C} = \{y : \delta'(x_0, y) = 0\}. \tag{2.8}$$

Note that the distance function $\delta(\cdot)$ and then the directional derivative $\delta'(x_0, \cdot)$ are Lipschitz and hence are continuous functions. Consequently, $\mathcal{C}$ is a closed cone. Now we show that S is approximated at x_0 by $\mathcal{C}$. Since $\delta(\cdot)$ is Lipschitz, the directional derivative $\delta'(x_0, y)$ gives the first-order approximation of $\delta(\cdot)$ uniformly in y, i.e.

$$\delta(x_0 + y) = \delta'(x_0, y) + o(\|y\|) \tag{2.9}$$

(e.g., Demyanov and Rubinov [**4**, Lemma 3.2]). Then (2.2) follows immediately from (2.8) and (2.9). Also, because of (2.9) and since $\delta(x_0 + y) = 0$ for $x_0 + y \in S$, we have that

$$\delta'(x_0, y)/\|y\| = \delta'(x_0, y/\|y\|) \to 0$$

as $y \to 0$ and $x_0 + y \in S$. Since the function $d(\cdot, \mathcal{C})$ is continuous, this implies by the standard argument of compactness that $d(y/\|y\|, \mathcal{C}) \to 0$ as $y \to 0$ and $x_0 + y \in S$. Consequently, $d(y, \mathcal{C}) = o(\|y\|)$ for $x_0 + y \in S$ and (2.1) follows. □

It follows from Theorem 1 that if the approximating cone exists, then it can be represented in the form (2.8) and consequently is unique. Moreover, since all norms in $\mathbf{R}^n$ are equivalent, the definition of approximating cones is independent of a particular choice of the norm $\|\cdot\|$. Therefore, Theorem 1 implies the following result:

COROLLARY 1. *If for a given set S and a certain norm the distance function is directionally differentiable at $x \in S$, then it is directionally differentiable at x for any other norm.*

If the set S is convex, then the distance function $d(\cdot, S)$ is also convex and hence is directionally differentiable (e.g. [**9**]). Consequently, in this case the approximating cone exists at every $x \in S$ and coincides with the tangent cone cl $\bigcup\{\lambda(S - x) : \lambda > 0\}$.

Now we formulate the main result of this paper. With the cone $\mathcal{C}$ and norm $\|\cdot\|$ is associated the set-valued metric projection $\Pi\colon \mathbf{R}^n \rightrightarrows \mathcal{C}$,

$$\Pi y = \{z \in \mathcal{C} \colon \|y - z\| = d(y, \mathcal{C})\}.$$

One can see that for every y the set Πy is nonempty and compact and that the mapping Π is positively homogeneous, i.e. $\Pi(ty) = t\Pi y$ for all $t > 0$.

THEOREM 2. *Let S be approximated at $x_0 \in S$ by a cone $\mathcal{C}$. Then for every $y \neq 0$,*

$$\lim_{t \to 0^+} \frac{d(P_S(x_0 + ty), x_0 + t\Pi y)}{t} = 0. \tag{2.10}$$

PROOF. Suppose that (2.10) does not hold. Then there exists a vector $\bar{y} \neq 0$ and a sequence $t_n \to 0^+$ of positive numbers such that

$$d(z_n, t_n \Pi \bar{y}) \geq \varepsilon t_n, \tag{2.11}$$

where $z_n = P_S(x_0 + t_n \bar{y}) - x_0$ and ε is a positive constant. As in (2.4) we have that $\|z_n\| \leq 2\|t_n \bar{y}\|$ and hence by the argument of compactness the sequence $\{t_n\}$

can be chosen in such a way that $t_n^{-1} z_n$ converges to a vector $\bar{z}$, say. Note that it is implied by the condition (2.1) of the definition of approximating cones that $\bar{z} \in \mathcal{C}$. It follows from (2.11) that $d(t_n^{-1} z_n, \Pi \bar{y}) \geq \varepsilon$, and hence by the continuity of $d(\cdot, \Pi \bar{y})$ that $d(\bar{z}, \Pi \bar{y})$ is greater than zero. Consequently $\bar{z}$ does not belong to the set $\Pi \bar{y}$.

On the other hand,

$$\begin{aligned} \|\bar{y} - \bar{z}\| &= t_n^{-1} \|t_n \bar{y} - t_n \bar{z}\| = t_n^{-1}(\|t_n \bar{y} - z_n\| + o(t_n)) \\ &= t_n^{-1}[d(x_0 + t_n \bar{y}, S) + o(t_n)]. \end{aligned}$$

Moreover, by Theorem 1 we have that $t_n^{-1} d(x_0 + t_n \bar{y}, S) \to d(\bar{y}, \mathcal{C})$ and hence $\|\bar{y} - \bar{z}\| = d(\bar{y}, \mathcal{C})$. This implies that $\bar{z} \in \Pi \bar{y}$, a contradiction. □

It follows from Theorem 2 that if the approximating cone $\mathcal{C}$ exists and $\Pi \bar{y} = \{\bar{z}\}$ is a singleton for some $\bar{y} \in \mathbf{R}^n$, then

$$P_S(x_0 + t\bar{y}) = x_0 + t\bar{z} + o(t), \qquad t > 0.$$

Thus $\bar{z}$ is the directional derivative of P_S in the direction $\bar{y}$. In this sense existence of the approximating cone is a sufficient condition for directional differentiability of P_S. On the other hand, it is clear that if P_S is directionally differentiable at $x_0 \in S$, then the distance function $d(\cdot, S)$ also is, and hence the approximating cone exists. Consequently, existence of the approximating cone is always a necessary condition for directional differentiability of P_S.

3. Cone approximation of a set defined by inequality constraints. The concept of approximating cones closely resembles the standard notion of Bouligand's contingent cone $T_S(x_0)$, which can be defined as follows:

$$T_S(x_0) = \left\{ y \colon \liminf_{t \to 0^+} \frac{d(x_0 + ty, S)}{t} = 0 \right\}.$$

(For an equivalent definition and elementary properties of $T_S(x_0)$ see, e.g., [**1**, pp. 176–179].) Theorem 1 implies that whenever the approximating cone exists it coincides with the contingent cone $T_S(x_0)$. In the following example we demonstrate the difference between contingent and approximating cones.

EXAMPLE. Consider the set $S = \{2^{-k} \colon k = 1, 2, \dots\} \cup \{0\}$ in $\mathbf{R}$. Then $T_S(0) = \mathbf{R}_+$, whereas the distance function $d(\cdot, S)$ is not differentiable at zero in the positive direction, and hence the approximating cone does not exist at $x_0 = 0$.

In the remainder of this section we study the case where S is defined by an inequality constraint as follows:

$$S = \{x \colon g(x) \leq 0\}.$$

We suppose that the constraint function $g(x)$ is uniformly directionally differentiable at $x_0 \in S$, i.e. $g(x_0 + y) - g(x_0) = g'(x_0, y) + o(\|y\|)$, and the directional derivative $g'(x_0, y)$ is continuous in y. This holds, for example, if $g(x)$ is directionally differentiable and locally Lipschitz. Let $g(x_0) = 0$ and consider the cone

$$\mathcal{C} = \{y : g'(x_0, y) \leq 0\}. \tag{3.1}$$

We say that the *nondegeneracy condition* holds at x_0 if

$$\mathcal{C} = \operatorname{cl}\{y : g'(x_0, y) < 0\}.$$

We remark here that if $g(x)$ is Fréchet differentiable at x_0 (with nonzero gradient) and hence $g'(x_0, y)$ is linear in y, then the cone $\mathcal{C}$ is a half-space and the nondegeneracy condition holds automatically.

THEOREM 3. *Suppose that the nondegeneracy condition holds at the point x_0. Then S is approximated at x_0 by the cone $\mathcal{C}$.*

PROOF. We have to show that conditions (2.1) and (2.2) are satisfied. Suppose that condition (2.1) does not hold. Then there exists a sequence $\{x_n\}$ in S converging to x_0 such that $y_n/\|y_n\|$, with $y_n = x_n - x_0$, tends to a vector $\overline{y} \notin \mathcal{C}$. But

$$0 \geq g(x_0 + y_n)/\|y_n\| = g'(x_0, y_n/\|y_n\|) + o(\|y_n\|)/\|y_n\|.$$

Consequently, we obtain from the continuity of $g'(x_0, \cdot)$ that $g'(x_0, \overline{y}) \leq 0$, a contradiction.

Now suppose that (2.2) does not hold. Then there exists a sequence $\{y_n\}$ in $\mathcal{C}$ converging to zero such that $d(x_0 + y_n, S) \geq 2\varepsilon\|y_n\|$ for some $\varepsilon > 0$ and $y_n/\|y_n\|$ tends to a vector $\overline{y}$. Since $\mathcal{C}$ is closed, $\overline{y} \in \mathcal{C}$. Moreover, because of the nondegeneracy condition there exists a vector y^* such that $g'(x_0, y^*) < 0$, $\|y^*\| = \|\overline{y}\|$ and $\|y^* - \overline{y}\| < \varepsilon$. Consider the sequence $y_n^* = y_n + \|y_n\|(y^* - \overline{y})$. Then

$$d(x_0 + y_n^*, S) \geq d(x_0 + y_n, S) - \|y_n\|\,\|y^* - \overline{y}\| \geq \varepsilon\|y_n\|$$

and $\|y_n^*\|/\|y_n\| \to 1$. Consequently, $x_0 + y_n^*$ does not belong to S for all n and $y_n^*/\|y_n^*\|$ tends to y^*. From the continuity of $g'(x_0, \cdot)$ we have that $g'(x_0, y_n^*/\|y_n^*\|) < -\delta$ for some $\delta > 0$ and hence $g'(x_0, y_n^*) < -\delta\|y_n^*\|$. Moreover, since $g(x_0) \leq 0$ and because of the uniform directional differentiability of $g(x)$,

$$g(x_0 + y_n^*) \leq g'(x_0, y_n^*) + o(\|y_n^*\|) \leq -\delta\|y_n^*\| + o(\|y_n^*\|).$$

Consequently, $g(x_0 + y_n^*) \leq 0$ and hence $x_0 + y_n^* \in S$ for sufficiently large n, a contradiction. □

Now let the set S be defined by a (possibly infinite) number of inequality constraints as follows:

$$S = \{x \colon g_\alpha(x) \leq 0, \alpha \in \Lambda\}.$$

Then we can replace these inequality constraints by one constraint $g(x) \leq 0$ with g being the max-function.

$$g(x) = \sup\{g_\alpha(x) \colon \alpha \in \Lambda\}.$$

By the well-known theorem on Danskin [**3**] we have that if

(a) The index set Λ is compact,

(b) $g_\alpha(\cdot)$ is continuously differentiable for all $\alpha \in \Lambda$,

(c) $g(x, \alpha) = g_\alpha(x)$ together with $\nabla_x g(x, \alpha)$ are continuous on $\mathbf{R}^n \times \Lambda$.

Then $g(x)$ is uniformly directionally differentiable and

$$g'(x_0, y) = \max\{y^T \nabla g_\alpha(x_0) \colon \alpha \in \Lambda^*(x_0)\}, \tag{3.2}$$

where $\Lambda^*(x) = \{\alpha \in \Lambda \colon g(x) = g_\alpha(x)\}$.

In this case the nondegeneracy condition means that

(d) There exists a vector b such that $b^T \nabla g_\alpha(x_0) < 0$ for all $\alpha \in \Lambda^*(x_0)$ (i.e. the Mangasarian-Fromovitz constraint qualification [**6**]).

Here the directional derivative $g'(x_0, \cdot)$ is representable as the pointwise maximum (3.2) of a family of linear functions and hence is convex. Consequently, the cone $\mathcal{C}$ given in (3.1) is convex. Therefore we obtain that if conditions (a)–(d) are satisfied and the norm $\|\cdot\|$ is strictly convex, then P_S is directionally differentiable at the point x_0.

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