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A dynamic programming approach to adjustable robust optimization

Alexander Shapiro

School of Industrial & Systems Engineering, Georgia Institute of Technology, 765 Ferst Drive, Atlanta, GA 30332, United States

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ABSTRACT

In this paper we consider the adjustable robust approach to multistage optimization, for which we derive dynamic programming equations. We also discuss this from the point of view of risk averse stochastic programming. We consider as an example a robust formulation of the classical inventory model and show that, like for the risk neutral case, a basestock policy is optimal.

1. Introduction

In the last ten years, robust optimization has become a viable tool in dealing with optimization under uncertainty. For an excellent overview of the state of the art in that area of research we refer the reader to the recent book by Ben-Tal et al. [5]. In a dynamical setting, an adjustable approach to robust optimization, as an alternative to multistage stochastic programming, was introduced in [3] and is discussed in detail in [5, Chapter 14]. Although in various forms robust formulations of multistage problems were considered in several publications (e.g., [3,6,9,16]), it seems that a connection between dynamic programming equations and optimization over feasible policies was not clearly stated. Closest to our derivations may be an approach presented in [10].

We also discuss the adjustable robust approach to multistage optimization in a framework of risk averse stochastic programming. To this end we use the methodology of coherent risk measures and conditional risk mappings. The term "risk measure" could be somewhat misleading. Some authors use the term "acceptability functionals" (e.g., [11]). Anyway the terminology of "risk measures" became standard, so we will follow it here. Axioms of coherent risk measures were introduced in [2] and their theory is thoroughly developed in [7]; conditional risk mappings were discussed in [13,8,16]. In particular, we show how dynamic programming equations can be naturally written for adjustable multistage robust optimization. We discuss as an example robust formulation of the classical inventory model and show that, like in the risk neutral case (see, e.g., [19]), a basestock policy is optimal.

E-mail address: ashapiro@isye.gatech.edu.

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This paper is organized as follows. In the next section we give a general definition of a class of robust multistage problems and show how dynamic programming equations can be written for such problems. It seems that in all generality, when the uncertainty set is not necessarily the direct (Cartesian) product of stagewise uncertainty sets, these dynamic programming equations have not been explicitly written out before. In Section 3 we show that similar results can be derived in a general framework of dynamic risk measures. The main point of that section is Eq. (3.7) (see also (3.8)) which states that the composition of sup-risk measures is again the corresponding sup-risk measure. Although the material of Section 2 can be considered from the point of view of Section 3, there are two reasons for writing it out explicitly. First, Section 2 is self-contained with a rather elementary presentation, while derivations of Section 3 are indirect and based on sophisticated concepts from probability theory and functional analysis. Second, and maybe more importantly, by the nature of the material the essential sup-measures considered in Section 3 are defined up to sets of measure zero which is unnatural from the robustness point of view. Finally, in Section 4 we discuss an example of a dynamic inventory model.

2. Multistage robust optimization

Consider the following robust formulation of the multistage problem:

$$\begin{array}{ll} \underset{x_1,x_2(\cdot),\dots,x_T(\cdot)}{\operatorname{Min}} & \sup_{(\xi_2,\dots,\xi_T)\in\mathfrak{D}} \left[f_1(x_1)+f_2(x_2,\xi_2)+\cdots+f_T(x_T,\xi_T)\right] \\ \text{s.t.} & x_1\in\mathcal{X}_1,\quad x_t\in\mathcal{X}_t(x_{t-1},\xi_t),\quad t=2,\dots,T. \end{array} \tag{2.1}$$

Here $\xi_t \in \mathbb{R}^{d_t}$, $t = 2, \dots, T$, are data vectors (uncertainty parameters), $\mathfrak{D} \subset \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_T}$ is the uncertainty set, $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}$

and $f_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}$, $t = 2, \dots, T$, are objective functions, and $\mathcal{X}_1 \subset \mathbb{R}^{n_1}$ and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \mathbb{R}^{d_t} \rightrightarrows \mathbb{R}^{n_t}$ are multifunctions (point-to-set mappings) representing feasibility sets. We denote by $\xi_{[t]} = (\xi_2, \dots, \xi_t)$ the history of the data process up to time $t = 2, \dots, T$. The optimization (minimization) in (2.1) is performed over feasible policies (also called decision rules). Recall that a policy is a sequence of functions $x_1, x_2(\xi_{[2]}), \dots, x_T(\xi_{[T]})$, which is feasible if

$$x_1 \in \mathcal{X}_1, \qquad x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t),$$
$$\forall (\xi_2, \dots, \xi_T) \in \mathfrak{D}, \quad t = 2, \dots, T. \tag{2.2}$$

For $t \geq 2$ the decisions $x_t = x_t(\xi_{[t]})$ are functions of the data process, up to time t, i.e., are of a wait-and-see type called *adjustable* in robust optimization (cf. [5]). For $t = 1$ the notation $x_1(\xi_{[1]})$ stands for deterministic vector $x_1 \in \mathbb{R}^{n_1}$.

Let us consider the following construction. Denote by $\mathcal{Z}_t$, $t = 2, \dots, T$, the linear space of bounded real valued functions $Z : \mathbb{R}^{d_2} \times \dots \times \mathbb{R}^{d_t} \to \mathbb{R}$, with $\mathcal{Z}_1 \equiv \mathbb{R}$ (i.e., $\mathcal{Z}_1$ is the space of constants). For $1 \leq s < t \leq T$ consider the mapping $\rho_{t|s} : \mathcal{Z}_t \to \mathcal{Z}_s$ defined as follows:

$$[\rho_{t|s}(Z)](\xi_{[s]}) = \sup_{(\xi'_2, \dots, \xi'_T) \in \mathfrak{D}} \left\{ Z(\xi'_{[t]}) : \xi'_{[s]} = \xi_{[s]} \right\}, \quad Z \in \mathcal{Z}_t. \tag{2.3}$$

In particular, $\rho_{t|1} : \mathcal{Z}_t \to \mathbb{R}$ and

$$\rho_{t|1}(Z) = \sup_{(\xi_2, \dots, \xi_T) \in \mathfrak{D}} Z(\xi_{[t]}). \tag{2.4}$$

Note that the objective function in the right hand side of (2.3) does not depend on $\xi'_{t+1}, \dots, \xi'_T$ and the maximization can be performed over the set $\mathfrak{D}_t$ (instead of $\mathfrak{D}$), where $\mathfrak{D}_t$ is the projection of $\mathfrak{D}$ onto $\mathbb{R}^{d_2} \times \dots \times \mathbb{R}^{d_t}$, i.e.,

$$\mathfrak{D}_t = \left\{ \xi_{[t]} : \exists \xi'_{[T]} \in \mathfrak{D} \text{ such that } \xi_{[t]} = \xi'_{[t]} \right\}. \tag{2.5}$$

For $t = T, \dots, 2$, consider the following dynamic programming equations:

$$Q_t(x_{t-1}, \xi_{[t]}) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) \right\}, \tag{2.6}$$

where

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) = \rho_{t+1|t}\left[Q_{t+1}(x_t, \xi_{[t+1]}) \right], \tag{2.7}$$

with $\mathcal{Q}_{T+1}(\cdot, \cdot) \equiv 0$ by definition. At the first stage we need to solve the problem

$$\operatorname*{Min}_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathcal{Q}_2(x_1). \tag{2.8}$$

We are going to establish a connection between these dynamic equations and the multistage robust problem (2.1). The mapping $\rho_{t+1|t}$ in the right hand side of (2.7) is applied to the function $Q_{t+1}(x_t, \cdot)$ for given (fixed) x_t. That is, the *cost-to-go functions*, defined in (2.7), can be written as

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) = \sup_{(\xi'_2, \dots, \xi'_T) \in \mathfrak{D}} \left\{ Q_{t+1}\left(x_t, \xi'_{[t+1]}\right) : \xi'_{[t]} = \xi_{[t]} \right\}. \tag{2.9}$$

Of course, in order for the function $\mathcal{Q}_{t+1}(x_t, \xi_{[t]})$ to be real valued we need to impose some boundedness conditions ensuring that the maximum in the right hand side of (2.9) is finite.

It immediately follows from the definition (2.3) that for $1 \leq r < s < t \leq T$, the composite mapping $\rho_{s|r} \circ \rho_{t|s} : \mathcal{Z}_t \to \mathcal{Z}_r$ coincides with the mapping $\rho_{t|r} : \mathcal{Z}_t \to \mathcal{Z}_r$, i.e.,

$$\rho_{s|r} \circ \rho_{t|s} = \rho_{t|r}. \tag{2.10}$$

We also will need the following interchangeability property. Let A and B be two (abstract) nonempty sets, $A \ni x \mapsto \mathcal{B}(x) \subset B$ be a point-to-set mapping and $h : A \times B \to \mathbb{R}$ be a real valued function. Consider the min–max problem

$$\operatorname*{Max}_{x \in A} \inf_{y \in \mathcal{B}(x)} h(x, y). \tag{2.11}$$

Let $\mathcal{Y}$ be the space of mappings $y(\cdot) : A \to B$ such that $y(x) \in \mathcal{B}(x)$ for all $x \in A$, and consider the problem

$$\operatorname*{Min}_{y(\cdot) \in \mathcal{Y}} \sup_{x \in A} h(x, y(x)). \tag{2.12}$$

Proposition 2.1. *Suppose that* $\inf_{y \in \mathcal{B}(x)} h(x, y)$ *is finite for every* $x \in A$. *Then the optimal values of problems* (2.11) *and* (2.12) *are equal to each other. Moreover,* $\bar{y}(\cdot) \in \mathcal{Y}$ *is an optimal solution of problem* (2.12) *if*

$$\bar{y}(x) \in \arg \min_{y \in \mathcal{B}(x)} h(x, y), \quad \forall x \in A. \tag{2.13}$$

Proof. For any $y(\cdot) \in \mathcal{Y}$ we have that $h(x, y(x)) \geq \inf_{y \in \mathcal{B}(x)} h(x, y)$ for any $x \in A$, and hence

$$\sup_{x \in A} h(x, y(x)) \geq \sup_{x \in A} \inf_{y \in \mathcal{B}(x)} h(x, y).$$

It follows that the optimal value of problem (2.11) is less than or equal to the optimal value of problem (2.12).

Conversely, for a chosen $\varepsilon > 0$ let $\bar{y}(\cdot) \in \mathcal{Y}$ be such that

$$\inf_{y \in \mathcal{B}(x)} h(x, y) \geq h(x, \bar{y}(x)) - \varepsilon, \quad x \in A. \tag{2.14}$$

Such a mapping exists since it is assumed that $\inf_{y \in \mathcal{B}(x)} h(x, y)$ is finite (in particular the set $\mathcal{B}(x)$ is nonempty) for every $x \in A$. It follows that

$$\sup_{x \in A} \inf_{y \in \mathcal{B}(x)} h(x, y) \geq \sup_{x \in A} h(x, \bar{y}(x)) - \varepsilon, \tag{2.15}$$

and hence

$$\sup_{x \in A} \inf_{y \in \mathcal{B}(x)} h(x, y) \geq \inf_{y(\cdot) \in \mathcal{Y}} \sup_{x \in A} h(x, y(x)) - \varepsilon. \tag{2.16}$$

Since $\varepsilon > 0$ is arbitrary, it follows that the optimal value of problem (2.12) is less than or equal to the optimal value of problem (2.11).

Moreover, $\bar{y}(\cdot)$ is an optimal solution of (2.12) iff $\varepsilon = 0$ in (2.15). In turn this holds if $\varepsilon = 0$ in (2.11), i.e., if (2.13) holds. □

Suppose for the moment that $\mathcal{B}(x) = B$ for all $x \in A$ and that problem (2.12) attains its maximal value at a constant mapping $y(x) \equiv \bar{y}$. Then

$$\sup_{x \in A} \inf_{y \in B} h(x, y) = \inf_{y \in B} \sup_{x \in A} h(x, y). \tag{2.17}$$

Moreover, if $\bar{x} \in A$ is an optimal solution of problem (2.11), then $(\bar{x}, \bar{y})$ is a saddle point of problem (2.11). Conversely, if $(\bar{x}, \bar{y})$ is a saddle point of problem (2.11), then $\bar{x}$ is an optimal solution of problem (2.11) and $y(\cdot) \equiv \bar{y}$ is an optimal solution of problem (2.12).

The interchangeability property discussed in the above proposition is not new of course. In different contexts variants of this property were used by many authors; its origins can be traced to von Neumann's minimax theory.

Consider now the multistage problem (2.1). Recall that the minimization is performed over policies satisfying feasibility constraints (2.2). For fixed (feasible) decisions $x_1, x_2(\cdot), \dots, x_{T-1}(\cdot)$, let us consider minimization with respect to $x_T(\cdot)$. Assuming that the cost-to-go functions are finite valued, by Proposition 2.1 we can interchange the corresponding minimization and maximization in (2.1). This results in the problem

$$\begin{array}{c}\underset{x_1,x_2(\cdot),\dots,x_{T-1}(\cdot)}{\text{Min}} \ \sup_{\xi_{[T]}\in\mathfrak{D}} \\ \left[f_1(x_1)+\cdots+f_{T-1}(x_{T-1},\xi_{T-1})+\underbrace{\inf_{x_T\in\mathcal{X}_T(x_{T-1},\xi_T)} f_T(x_T,\xi_T)}_{Q_T(x_{T-1},\xi_T)} \right] \\ \text{s.t.}\ \ x_1\in\mathcal{X}_1,\ x_t\in\mathcal{X}_t(x_{t-1},\xi_t),\ t=2,\dots,T-1. \end{array} \tag{2.18}$$

Performing maximization in (2.18) with respect to ξ_T we can write (2.18) as

$$\begin{array}{c}\underset{x_1,x_2(\cdot),\dots,x_{T-1}(\cdot)}{\text{Min}} \ \sup_{\xi_{[T]}\in\mathfrak{D}} \\ \left[f_1(x_1)+\cdots+f_{T-1}(x_{T-1},\xi_{T-1})+\underbrace{\sup_{\xi_{[T]}\in\mathfrak{D}} Q_T(x_{T-1},\xi_T)}_{\mathcal{Q}_T(x_{T-1},\xi_{[T-1]})} \right] \\ \text{s.t.}\ \ x_1\in\mathcal{X}_1,\ x_t\in\mathcal{X}_t(x_{t-1},\xi_t),\ t=2,\dots,T-1. \end{array} \tag{2.19}$$

Note that the objective function in (2.19) does not depend on ξ_T and the maximization can be performed over $\xi_{[T-1]}\in\mathfrak{D}_{T-1}$ instead of $\xi_{[T]}\in\mathfrak{D}$.

Next we can proceed to minimization in (2.19) with respect to $x_{T-1}(\cdot)$. Again using the interchangeability property we obtain

$$\begin{array}{c}\underset{x_1,x_2(\cdot),\dots,x_{T-2}(\cdot)}{\text{Min}} \ \sup_{\xi_{[T-1]}\in\mathfrak{D}_{T-1}} \\ \left[f_1(x_1)+\cdots+\underbrace{\inf_{x_{T-1}\in\mathcal{X}_{T-1}(x_{T-2},\xi_{T-1})} \left\{ f_{T-1}(x_{T-1},\xi_{T-1})+\mathcal{Q}_T(x_{T-1},\xi_{[T-1]})\right\}}_{Q_{T-1}(x_{T-2},\xi_{[T-1]})} \right] \\ \text{s.t.}\ \ x_1\in\mathcal{X}_1,\ x_t\in\mathcal{X}_t(x_{t-1},\xi_t),\ t=2,\dots,T-2. \end{array}$$

Furthermore, by taking the maximum in the above problem with respect to ξ_{T-1} we obtain

$$\begin{array}{c}\underset{x_1,x_2(\cdot),\dots,x_{T-2}(\cdot)}{\text{Min}} \ \sup_{\xi_{[T-2]}\in\mathfrak{D}_{T-2}} \\ \left[f_1(x_1)+\cdots+f_{T-2}(x_{T-2},\xi_{T-2})+\mathcal{Q}_{T-1}(x_{T-2},\xi_{[T-2]})\right] \\ \text{s.t.}\ \ x_1\in\mathcal{X}_1,\ x_t\in\mathcal{X}_t(x_{t-1},\xi_t),\ t=2,\dots,T-2. \end{array} \tag{2.20}$$

Continuing this process backwards in time we derive dynamic equations (2.6)–(2.7). This gives the following result.

Proposition 2.2. *Suppose that the cost-to-go functions in dynamic equations (2.6)–(2.8) are finite valued. Then the optimal value of problem (2.1) is equal to the optimal value of problem (2.8). Moreover, a policy $\bar{x}_t(\xi_{[t]})$, $t=1,\dots,T$, is optimal for the problem (2.1) if*

$$\bar{x}_t(\xi_{[t]})\in\arg\min_{x_t\in\mathcal{X}_t(\bar{x}_{t-1}(\xi_{[t-1]}),\xi_t)}\left\{f_t(x_t,\xi_t)+\mathcal{Q}_{t+1}(x_t,\xi_{[t]})\right\},$$
$$t=2,\dots,T, \tag{2.21}$$

and $\bar{x}_1$ is an optimal solution of the first-stage problem (2.8).

Suppose for the moment that the uncertainty set is the direct product of nonempty sets $\mathcal{D}_t\subset\mathbb{R}^{d_t}$, $t=2,\dots,T$, i.e., $\mathfrak{D}=\mathcal{D}_2\times\cdots\times\mathcal{D}_T$. In that case the max-mapping $\rho_{t|s}$ takes the form

$$[\rho_{t|s}(Z)](\xi_{[s]})=\sup_{\xi'_{s+1}\in\mathcal{D}_{s+1},\dots,\xi'_t\in\mathcal{D}_t}\left\{Z(\xi_2,\dots,\xi_s,\xi'_{s+1},\dots,\xi'_t)\right\}. \tag{2.22}$$

Consequently, in that case dynamic equations (2.6)–(2.7) become

$$Q_t(x_{t-1},\xi_t)=\inf_{x_t\in\mathcal{X}_t(x_{t-1},\xi_t)}\left\{f_t(x_t,\xi_t)+\mathcal{Q}_{t+1}(x_t)\right\},$$
$$t=T,\dots,2, \tag{2.23}$$

with cost-to-go functions

$$\mathcal{Q}_{t+1}(x_t)=\sup_{\xi_{t+1}\in\mathcal{D}_{t+1}}Q_{t+1}(x_t,\xi_{t+1}) \tag{2.24}$$

independent of the data process.

3. Sup-risk measures

In a sense, derivations of Section 2 can be considered in the framework of dynamical risk measures. In this section we briefly outline the connection. Consider space $\mathcal{Z}=L_\infty(\Omega,\mathcal{F},P)$ of essentially bounded measurable functions $Z:\Omega\to\mathbb{R}$ and define

$$\rho^{\max}(Z)=\operatorname*{ess\,sup}_{\omega\in\Omega}Z(\omega),\quad Z\in\mathcal{Z}. \tag{3.1}$$

Recall that

$$\operatorname*{ess\,sup}_{\omega\in\Omega}Z(\omega)=\inf\left\{\sup_{\omega\in\Omega}Z'(\omega):Z'(\omega)=Z(\omega)\text{ a.e. }\omega\in\Omega\right\}. \tag{3.2}$$

We assume that Ω is a closed subset of $\mathbb{R}^d$ equipped with its Borel sigma algebra $\mathcal{F}$, P is a probability measure on $(\Omega,\mathcal{F})$, and that the support of P coincides with Ω, i.e., for any closed set $A\subset\Omega$, $A\neq\Omega$, we have that $P(A)<1$. Unless stated otherwise all probabilistic statements will be made with respect to the reference probability measure P.

The function $\rho^{\max}:\mathcal{Z}\to\mathbb{R}$ is a coherent risk measure (cf. [2]), i.e., it satisfies the conditions of convexity, monotonicity and positive homogeneity, and $\rho^{\max}(Z+a)=\rho^{\max}(Z)+a$ for any $Z\in\mathcal{Z}$ and $a\in\mathbb{R}$. We refer to $\rho^{\max}$ as the *sup-risk measure*. The sup-risk measure is law invariant, i.e., if $Z\in\mathcal{Z}$ and $Z'\in\mathcal{Z}$ have the same probability distribution, then $\rho^{\max}(Z)=\rho^{\max}(Z')$. It has the following dual representation (cf. [11, p. 55]):

$$\rho^{\max}(Z)=\sup_{\zeta\in\mathfrak{A}}\int_\Omega\zeta(\omega)Z(\omega)\mathrm{d}P(\omega), \tag{3.3}$$

where $\mathfrak{A}\subset L_1(\Omega,\mathcal{F},P)$ is the set of density functions, i.e., $\zeta\in L_1(\Omega,\mathcal{F},P)$ belongs to $\mathfrak{A}$ iff $\zeta(\omega)\geq 0$ for a.e. $\omega\in\Omega$ and $\int_\Omega\zeta(\omega)\mathrm{d}P(\omega)=1$. We also can write the dual representation (3.3) in the form

$$\rho^{\max}(Z)=\sup_{Q\in\mathfrak{Q}}\mathbb{E}_Q[Z], \tag{3.4}$$

where $\mathfrak{Q}$ denotes the set of probability measures on $(\Omega,\mathcal{F})$ absolutely continuous with respect to P. Recall that by the Radon–Nikodym theorem, $\mathfrak{Q}=\{Q:dQ=\zeta\,\mathrm{d}P,\ \zeta\in\mathfrak{A}\}$.

In [11, p. 54] the sup-risk measure was introduced as a limit of the average value-at-risk (also called the conditional value-at-risk [14])

$$\mathsf{AV@R}_\alpha(Z)=\inf_{t\in\mathbb{R}}\left\{t+\alpha^{-1}\mathbb{E}[Z-t]_+\right\}$$

as $\alpha\downarrow 0$, and hence was denoted as $\mathsf{AV@R}_0$. Note, however, that for $\alpha\in(0,1)$, the function $\mathsf{AV@R}_\alpha(\cdot)$ is naturally defined on the space $L_1(\Omega,\mathcal{F},P)$ since it is the largest of the spaces $L_p(\Omega,\mathcal{F},P)$, $p\in[1,\infty]$, on which it is real valued and continuous. On the other hand, the sup-function $\rho^{\max}(\cdot)$ is naturally defined on the space $L_\infty(\Omega,\mathcal{F},P)$, on which it is finite valued and continuous, while it will have $+\infty$ values on $L_p(\Omega,\mathcal{F},P)$ space for any $p\in[1,+\infty)$. Recall that $L_\infty(\Omega,\mathcal{F},P)$ is the dual of the Banach space $L_1(\Omega,\mathcal{F},P)$. It follows, e.g., from the dual representation (3.3), that $\rho^{\max}:L_\infty(\Omega,\mathcal{F},P)\to\mathbb{R}$ is lower semicontinuous in the weak* topology induced by the space $L_1(\Omega,\mathcal{F},P)$.

If we view set Ω as an uncertainty set, then the only difference between $\rho^{\max}$ and the max-function defined in (2.4) is with respect to the family of subsets of Ω of measure zero. Of course, if $Z(\cdot)$ is continuous on Ω, then $\rho^{\max}(Z)=\sup_{\omega\in\Omega}Z(\omega)$. The interchangeability property derived in Proposition 2.1 is similar and in a sense can be viewed as a particular case of the corresponding interchangeability property of coherent risk measures (cf. [15], [18, Proposition 6.37]).

In order to extend this to a dynamic (multistage) setting we need to extend the concept of sup-risk measure to a conditional risk mapping. Let $\mathcal{G}$ be a sigma subalgebra of $\mathcal{F}$ and $\mathcal{Z}' := L_\infty(\Omega, \mathcal{G}, P)$. A conditional risk mapping is a mapping $\rho : \mathcal{Z} \to \mathcal{Z}'$ satisfying conditions of convexity, monotonicity and positive homogeneity, and such that $\rho(Z + Y) = \rho(Z) + Y$ for any $Z \in \mathcal{Z}$ and $Y \in \mathcal{Z}'$ (cf. [13,16]). In particular, it follows that $\rho(Y) = Y$ for any $Y \in \mathcal{Z}'$.

It is possible to define a conditional version $\rho^{\max}_{|\mathcal{G}} : \mathcal{Z} \to \mathcal{Z}'$ of the sup-risk measure in several ways. For example, it can be viewed as a limit of conditional $\mathsf{AV@R}_\alpha(Z|\mathcal{G})$ as $\alpha \downarrow 0$ (cf. [11,12]). We can also look at the conditional sup-risk mappings from the following point of view. Consider a sequence of random variables $Y_i \in \mathcal{Z}$, $i = 1, \ldots,$ and denote by $\mathfrak{S}_t \subset \mathbb{R}^t$ the support of the distribution of random vector $(Y_1, \ldots, Y_t)$. For the sake of simplicity we consider here a sequence of random numbers $Y_1, \ldots, Y_t$; similar derivations can be performed of course for a sequence of random vectors. Note that since each Y_i is essentially bounded, the set $\mathfrak{S}_t$ is bounded in $\mathbb{R}^t$. Let $\mathcal{F}_t \subset \mathcal{F}$ be the sigma algebra generated by random vector $(Y_1, \ldots, Y_t)$ and let $\mathcal{Z}_t = L_\infty(\Omega, \mathcal{F}_t, P)$. We have that $Z \in \mathcal{Z}_t$ iff $Z = h(Y_1, \ldots, Y_t)$ for some measurable and bounded function $h(y)$. Then the conditional sup-risk mapping $\rho^{\max}_{\mathcal{F}_t|\mathcal{F}_{t-1}} : \mathcal{Z}_t \to \mathcal{Z}_{t-1}$ can be written as

$$\begin{aligned}\rho^{\max}_{\mathcal{F}_t|\mathcal{F}_{t-1}}(h(Y_1, \ldots, Y_t)) &= \operatorname{ess\,sup}\{h(y) : y \in \mathfrak{S}_t, y_1 \\ &= Y_1, \ldots, y_{t-1} = Y_{t-1}\}. \end{aligned} \tag{3.5}$$

The "ess sup" in (3.5) is understood with respect to the conditional distribution of Y_t given $(Y_1, \ldots, Y_{t-1}) = (y_1, \ldots, y_{t-1})$. That is,

$$\begin{aligned}&\operatorname{ess\,sup}\{h(y) : y \in \mathfrak{S}_t, y_1 = Y_1, \ldots, y_{t-1} = Y_{t-1}\} \\ &\quad = \inf\{x : \Pr(h(Y_1, \ldots, Y_t) > x | Y_1 = y_1, \ldots, Y_{t-1} \\ &\quad = y_{t-1}) = 0\}.\end{aligned}$$

Here the composite mapping $\rho^{\max}_{\mathcal{F}_{t-1}|\mathcal{F}_{t-2}} \circ \rho^{\max}_{\mathcal{F}_t|\mathcal{F}_{t-1}} : \mathcal{Z}_t \to \mathcal{Z}_{t-2}$ can be written as

$$\begin{aligned}\rho^{\max}_{\mathcal{F}_{t-1}|\mathcal{F}_{t-2}} \circ \rho^{\max}_{\mathcal{F}_t|\mathcal{F}_{t-1}} &= \operatorname{ess\,sup}\{h(y) : y \in \mathfrak{S}_t, y_1 \\ &= Y_1, \ldots, y_{t-2} = Y_{t-2}\},\end{aligned} \tag{3.6}$$

and hence the following analogue of formula (2.10) obviously holds:

$$\rho^{\max}_{\mathcal{F}_2|\mathcal{F}_1} \circ \rho^{\max}_{\mathcal{F}|\mathcal{F}_2} = \rho^{\max}_{\mathcal{F}|\mathcal{F}_1}. \tag{3.7}$$

In particular, the composite mapping $\rho^{\max}_{\mathcal{F}_1|\mathcal{F}_0} \circ \cdots \circ \rho^{\max}_{\mathcal{F}_t|\mathcal{F}_{t-1}} : \mathcal{Z}_t \to \mathbb{R}$, where $\mathcal{F}_0 := \{\emptyset, \Omega\}$, is

$$\begin{aligned}&\rho^{\max}_{\mathcal{F}_1|\mathcal{F}_0} \circ \cdots \circ \rho^{\max}_{\mathcal{F}_t|\mathcal{F}_{t-1}}(h(Y_1, \ldots, Y_t)) \\ &\quad = \operatorname{ess\,sup}\{h(y) : y \in \mathfrak{S}_t\}.\end{aligned} \tag{3.8}$$

That is, this composite mapping is the sup-risk measure on the space $\mathcal{Z}_t$.

It is possible now to use the general machinery of conditional risk mappings to write dynamic programming equations for the multistage problem considered (cf. [16,17], [18, Chapter 6]). The key observation here is that the composite mapping in the left hand side of (3.8) coincides with the corresponding sup-risk measure. Because of that the dynamic programming equations associated with the conditional sup-risk mappings are the same as the ones derived in Section 2 up to a change of the "ess sup" operators to "sup" operators.

4. The inventory model

Consider the following robust formulation of the inventory model (cf. [4, p. 254]):

$$\begin{aligned}&\operatorname*{Min}_{x_t \ge y_t} \sup_{d_{[T]} \in \mathfrak{D}} \left\{ \sum_{t=1}^{T} c_t(x_t - y_t) + \psi_t(x_t, d_t) \right\} \\ &\text{s.t.} \quad y_{t+1} = x_t - d_t, t = 1, \ldots, T-1.\end{aligned} \tag{4.1}$$

Here y_1 is a given initial inventory level, $d_1, \ldots, d_T$ is the demand process, $\mathfrak{D} \subset \mathbb{R}^T_+$ is the uncertainty set, c_t, b_t, h_t are the ordering, backorder penalty, and holding costs per unit, respectively, at time t, and

$$\psi_t(x_t, d_t) := b_t[d_t - x_t]_+ + h_t[x_t - d_t]_+.$$

We assume that $b_t > c_t > 0$ and $h_t \ge 0$, $t = 1, \ldots, T$, and that the uncertainty set $\mathfrak{D}$ is nonempty and bounded. Recall that the minimization in (4.1) is performed over feasible policies of the form $x_1, x_2(d_{[1]}), \ldots, x_T(d_{[T-1]})$. As before, we denote $(d_1, \ldots, d_t)$ by $d_{[t]}$. In particular, $d_{[T]} = (d_1, \ldots, d_T)$.

In accordance with derivations of Section 2, the dynamic programming equations for this problem can be written as follows. At the last stage $t = T$, for given (observed) inventory level y_T and given (observed) demand values $(d_1, \ldots, d_{T-1})$, we need to solve the problem

$$\operatorname*{Min}_{x_T \ge y_T} \left\{ c_T(x_T - y_T) + \sup_{(d_1, \ldots, d_T) \in \mathfrak{D}} \psi_T(x_T, d_T) \right\}. \tag{4.2}$$

The optimal value of problem (4.2) depends on y_T and $d_{[T-1]}$ and is denoted as $Q_T(y_T, d_{[T-1]})$. Continuing in this way, for $t = T-1, \ldots, 2$, the corresponding cost-to-go functions $Q_t(y_t, d_{[t-1]})$ are given as optimal values of the respective problems:

$$\begin{aligned}\operatorname*{Min}_{x_t \ge y_t} \Big\{ & c_t(x_t - y_t) + \sup_{d'_{[T]} \in \mathfrak{D}} \big[\psi_t(x_t, d'_t) \\ & + Q_{t+1}\big(x_t - d'_t, d'_{[t]}\big) : d'_{[t-1]} = d_{[t-1]} \big] \Big\}.\end{aligned} \tag{4.3}$$

Finally, at the first stage we need to solve the problem

$$\operatorname*{Min}_{x_1 \ge y_1} c_1(x_1 - y_1) + \sup_{d_{[T]} \in \mathfrak{D}} \big[\psi_1(x_1, d_1) + Q_2(x_1 - d_1, d_1) \big]. \tag{4.4}$$

It is straightforward to verify by induction in $t = T, \ldots,$ that the cost-to-go functions $Q_t(y_t, d_{[t-1]})$ are convex in y_t.

Suppose now that the uncertainty set $\mathfrak{D}$ is given by the direct product $\mathfrak{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_T$ for some nonempty bounded sets $\mathcal{D}_t \subset \mathbb{R}_+$, $t = 1, \ldots, T$; for example we can take $\mathcal{D}_t = [a_t, b_t]$ to be intervals. Then the cost-to-go function at the last stage is

$$Q_T(y_T) = \inf_{x_T \ge y_T} \left\{ c_T(x_T - y_T) + \sup_{d_T \in \mathcal{D}_T} \psi_T(x_T, d_T) \right\}. \tag{4.5}$$

And hence for $t = T-1, \ldots, 2$ the dynamic programming equations (4.3) can be written as (cf. [4, p. 254])

$$Q_t(y_t) = \inf_{x_t \ge y_t} \left\{ c_t(x_t - y_t) + \sup_{d_t \in \mathcal{D}_t} \big[\psi_t(x_t, d_t) + Q_{t+1}(x_t - d_t) \big] \right\}. \tag{4.6}$$

Note that here the cost-to-go function $Q_t(y_t)$, $t = 2, \ldots, T$, is independent of $d_{[t-1]}$, and is convex and continuous. Thus it follows by convexity arguments that a basestock policy $\bar{x}_t = \bar{x}_t(d_{[t-1]})$ is optimal (cf. [1]). Recall that a basestock policy is defined as $\bar{x}_t := \max\{y_t, x^*_t\}$, where x^*_t is an optimal solution of

$$\operatorname*{Min}_{x_t} \left\{ c_t x_t + \sup_{d_t \in \mathcal{D}_t} \big[\psi_t(x_t, d_t) + Q_{t+1}(x_t - d_t) \big] \right\}, \tag{4.7}$$

and $y_t = \bar{x}_{t-1} - d_{t-1}$, $t = 2, \ldots, T$, with y_1 being given.

It could be noted that if $\mathcal{D}_t = [a_t, b_t]$, $t = 1, \dots, T$, are intervals, then by convexity arguments the maximum in (4.6) is attained either at $\bar{d}_t = a_t$ or at $\bar{d}_t = b_t$. Therefore it suffices to consider the subset of the uncertainty set given by the direct product of sets $\{a_t, b_t\}$, $t = 1, \dots, T$. This subset has 2^T elements, and hence for not too large number of stages, say $T \le 20$, it is possible to solve the corresponding robust inventory model directly. This was the basis for numerical experiments of [4] (see also [5, Section 15.2.4.1]).

As another example, let the uncertainty set $\mathfrak{D}$ be given as

$$\mathfrak{D} := (\mathcal{D}_1 \times \cdots \times \mathcal{D}_T) \cap \{d \in \mathbb{R}^T : a^{\mathsf{T}} d \le b\},$$

for some $a \in \mathbb{R}^T$ and $b \in \mathbb{R}$. Suppose that the set $\mathfrak{D}$ is nonempty. Then the dynamic equations (4.3) take the form

$$Q_t(y_t, d_{[t-1]}) = \inf_{x_t \ge y_t} \left\{ c_t(x_t - y_t) + \sup_{\substack{d_t \in \mathcal{D}_t, \dots, d_T \in \mathcal{D}_T \\ a_1 D_1 + \cdots + a_{t-1} d_{t-1} + a_t d_t + \cdots + a_T d_T \le b}} \left[\psi_t(x_t, d_t) + Q_{t+1}\left(x_t - d_t, d_{[t]}\right) \right] \right\}. \tag{4.8}$$

In that case the cost-to-go function $Q_t(y_t, d_{[t-1]})$ depends only on y_t and $W_{t-1} := a_1 d_1 + \cdots + a_{t-1} d_{t-1}$. In these variables, Eq. (4.8) can be written as

$$Q_t(y_t, W_{t-1}) = \inf_{x_t \ge y_t} \left\{ c_t(x_t - y_t) + \sup_{\substack{d_t \in \mathcal{D}_t, \dots, d_T \in \mathcal{D}_T \\ a_t d_t + \cdots + a_T d_T \le b - W_{t-1}}} \left[\psi_t(x_t, d_t) + Q_{t+1}(x_t - d_t, W_{t-1} + a_t d_t) \right] \right\}. \tag{4.9}$$

Note that the cost-to-go functions $Q_t(y_t, W_{t-1})$ are defined only for those W_{t-1} for which the constraints in the maximization problem in the right hand side of (4.9) are feasible.

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References

[1] S. Ahmed, U. Cakmak, A. Shapiro, Coherent risk measures in inventory problems, European Journal of Operational Research 182 (2007) 226–238.
[2] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent measures of risk, Mathematical Finance 9 (1999) 203–228.
[3] A. Ben-Tal, A. Goryashko, A. Guslitzer, A. Nemirovski, Adjustable robust solutions of uncertain linear programs, Mathematical Programming 99 (2004) 351–376.
[4] A. Ben-Tal, B. Golany, A. Nemirovski, J.P. Vial, Retailer–supplier flexible commitments contracts: a robust optimization approach, Manufacturing & Service Operations Management 7 (2005) 248–271.
[5] A. Ben-Tal, L. El Ghaoui, A. Nemirovski, Robust Optimization, Princeton University Press, Princeton, 2009.
[6] D. Bertsimas, A. Thiele, A robust optimization approach to inventory theory, Operations Research 54 (2006) 150–168.
[7] H. Föllmer, A. Schield, Stochastic Finance: An Introduction In Discrete Time, in: De Gruyter Studies in Mathematics, 2nd ed., Berlin, 2004.
[8] M. Fritelli, E. Rosazza Gianin, Dynamic convex risk measures, in: G. Szegö (Ed.), Risk Measures for the 21st Century, Wiley, Chichester, 2004, pp. 227–248.
[9] G.N. Iyengar, Robust dynamic programming, Mathematics of Operations Research 30 (2005) 257–280.
[10] A. Nilim, L. El Ghaoui, Robust control of Markov decision processes with uncertain transition matrices, Operations Research 53 (2005) 780–798.
[11] G.Ch. Pflug, W. Römisch, Modeling, Measuring and Managing Risk, World Scientific, Singapore, 2007.
[12] G.Ch. Pflug, W. Römisch, The role of information in multi-period risk measurement, SPEPS, 2009.
[13] F. Riedel, Dynamic coherent risk measures, Stochastic Processes and Their Applications 112 (2004) 185–200.
[14] R.T. Rockafellar, S. Uryasev, Optimization of conditional value at risk, The Journal of Risk 2 (2000) 21–41.
[15] A. Ruszczyński, A. Shapiro, Optimization of convex risk functions, Mathematics of Operations Research 31 (2006) 433–452.
[16] A. Ruszczyński, A. Shapiro, Conditional risk mappings, Mathematics of Operations Research 31 (2006) 544–561.
[17] A. Ruszczyński, Risk-averse dynamic programming for Markov decision processes, Mathematical Programming, Ser. B 125 (2010) 235–261.
[18] A. Shapiro, D. Dentcheva, A. Ruszczyński, Lectures on Stochastic Programming: Modeling and Theory, SIAM, Philadelphia, 2009.
[19] P.H. Zipkin, Foundations of Inventory Management, McGraw-Hill, 2000.