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Time consistency of dynamic risk measures

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ABSTRACT

In this paper we discuss time consistency of risk averse multistage stochastic programming problems. We show, in a framework of finite scenario trees, that composition of law invariant coherent risk measures can be law invariant only for the expectation or max-risk measures.

1. Introduction

Consider a decision process of the from

$$\text{decision } (x_1) \rightsquigarrow \text{observation } (\mathcal{F}_2) \rightsquigarrow \text{decision } (x_2) \rightsquigarrow \cdots \rightsquigarrow \text{observation}(\mathcal{F}_T) \rightsquigarrow \text{decision}(x_T). \quad (1.1)$$

Here $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T$ is a sequence of sigma algebras (filtration) defined on a measurable space $(\Omega, \mathcal{F})$, with $\mathcal{F}_1 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$, representing flow of information and $x_t \in \mathbb{R}^{n_t}$, $t = 1, \dots, T$, are decision vectors. The basic requirement of such decision process is the *nonanticipativity* constraint. That is, at time period $t \in \{1, \dots, T\}$ our decision x_t should only depend on information available at time t, and should not depend on future observations. This can be formulated by considering $x_t : \Omega \to \mathbb{R}^{n_t}$ as a mapping which is required to be $\mathcal{F}_t$ measurable. Since $\mathcal{F}_1 = \{\emptyset, \Omega\}$ is trivial, this implies that the first stage decision x_1 is deterministic. If the set Ω is finite this process can be represented as a scenario tree with every node of the tree representing the corresponding state of the process with its children nodes representing possible states at the next period of time (see, e.g., [9, section 6.7.1] for a detailed description of this construction).

So far we haven't introduced any optimality criteria for our decisions. One standard approach is to minimize the expected value of the total cost:

$$\text{Min } \mathbb{E}[J(\omega)] \text{ subject to } (x_1, x_2(\cdot), \dots, x_T(\cdot)) \in \mathfrak{X}. \quad (1.2)$$

Here $f_t : \mathbb{R}^{n_t} \times \Omega \to \mathbb{R}$ are cost functions, $J(\omega) := f_1(x_1) + f_2(x_2(\omega), \omega) + \cdots + f_T(x_T(\omega), \omega)$ and

$$\mathfrak{X} := \{(x_1, x_2(\cdot), \dots, x_T(\cdot)) : x_1 \in \mathcal{X}, x_2(\cdot) \in \mathcal{X}_2(x_1, \cdot), \dots, x_T(\cdot) \in \mathcal{X}_T(x_{T-1}(\cdot), \cdot)\}$$

is the set of feasibility constraints. The functions $f_t(x_t, \cdot)$ and multifunctions $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \Omega \rightrightarrows \mathbb{R}^{n_t}$ are $\mathcal{F}_t$-measurable, the expectation is computed according to a specified probability measure P on the measurable space $(\Omega, \mathcal{F})$, and the optimization (minimization) in (1.2) is performed over mappings $x_t(\cdot)$, $t = 1, \dots, T$, satisfying the nonanticipativity constraint. We refer to problem (1.2), where the cost minimization is performed "on average", as the *risk neutral* formulation.

In recent years considerable attention has been given to *risk averse* formulations of multistage stochastic programming problems (with $T \geq 3$). That is, for a specified risk measure $\rho : \mathcal{Z} \to \mathbb{R}$, the following problem is considered

$$\text{Min } \rho[J(\omega)] \text{ subject to } (x_1, x_2(\cdot), \dots, x_T(\cdot)) \in \mathfrak{X}. \quad (1.3)$$

Here ρ is a functional assigning to a random variable $Z \in \mathcal{Z}$ numerical value $\rho(Z)$, with $\mathcal{Z}$ being a linear space of allowable random variables. In various applications it makes sense to use the space $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, of random variables $Z : \Omega \to \mathbb{R}$ having finite p-th order moments. In the terminology introduced by Artzner et al. [1], risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is said to be *coherent* if it satisfies the following conditions (axioms):

(A1) *Monotonicity:* If $Z, Z' \in \mathcal{Z}$ and $Z \succeq Z'$, then $\rho(Z) \geq \rho(Z')$.
(A2) *Subadditivity:*

$$\rho(Z + Z') \leq \rho(Z) + \rho(Z')$$

for all $Z, Z' \in \mathcal{Z}$.

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(A3) *Translation Equivariance:* If $a \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\rho(Z+a) = \rho(Z) + a$.
(A4) *Positive Homogeneity:* If $t \ge 0$ and $Z \in \mathcal{Z}$, then $\rho(tZ) = t\rho(Z)$.

The notation $Z \succeq Z'$ means that $Z(\omega) \ge Z'(\omega)$ for a.e. $\omega \in \Omega$.

It is said that risk measure ρ is *law invariant* if $\rho(Z) = \rho(Z')$ for any two random variables $Z, Z' \in L_p(\Omega, \mathcal{F}, P)$ having the same distribution. An important example of a (nonlinear) law invariant coherent risk measure is the so-called Average Value-at-Risk, which can be defined as

$$\mathsf{AVaR}_\alpha(Z) := \inf_{t\in\mathbb{R}} \left\{ t + (1-\alpha)^{-1}\mathbb{E}[Z-t]_+ \right\}, \tag{1.4}$$

with $\mathcal{Z} := L_1(\Omega, \mathcal{F}, P)$ and $\alpha \in [0, 1)$. It can be noted that for $\alpha = 0$, the $\mathsf{AVaR}_0(\cdot)$ coincides with the expectation operator $\mathbb{E}(\cdot)$. For another extreme case of α tending to 1,

$$\lim_{\alpha\uparrow 1} \mathsf{AVaR}_\alpha(Z) = \operatorname{ess\,sup} Z.$$

That is, $\mathsf{AVaR}_1(\cdot)$ can be defined as the essential supremum operator defined on the space $\mathcal{Z} := L_\infty(\Omega, \mathcal{F}, P)$.

From the point of view of the dynamic decision process (1.1) it is natural to consider a conceptual requirement of *time consistency*. One approach is to define time consistency from the point of view of optimal policies (strategies). In this respect we can cite Carpentier et al. [3] were this approach is discussed and appropriate references are given: *"The sequence of optimization problems is said to be dynamically consistent if the optimal strategies obtained when solving the original problem at time t_1 remain optimal for all subsequent problems"*. A similar approach was suggested in [8]: *optimality of the decision at a state of the process at time $t \in \{1, \dots, T-1\}$ should not involve states which do not follow that state, i.e., cannot happen in the future.* If the decision process is represented by the corresponding scenario tree, this means that if at a time t we are at a certain node of the tree, then optimality of our future decisions should not depend on scenarios which do not pass through this node.

In order to formalize this concept of time consistency we need to define what do we optimize (say minimize) at every state of the process looking into the future. That is, in the scenario tree formulation, at every node at time $t \in \{1, \dots, T-1\}$ we should define a risk measure on the space of children nodes of that node, and hence construct the corresponding conditional risk mapping $\rho_{t+1|\mathcal{F}_t}$ (see [9, section 6.7.1] for a detailed description of such a construction). Recall that the sigma algebra $\mathcal{F}_1$ is trivial and hence $\rho_{2|\mathcal{F}_1}$ is a real valued risk measure. This will define a nested formulation of the associated optimization problem (see also [6]). This nested formulation can be written in the equivalent form

$$\operatorname{Min} \varrho[J(\omega)] \text{ subject to } (x_1, x_2(\cdot), \dots, x_T(\cdot)) \in \mathfrak{X}, \tag{1.5}$$

where $\varrho : \mathcal{Z} \to \mathbb{R}$ is the risk measure defined as the composition

$$\varrho(Z) = \rho_{2|\mathcal{F}_1}\left(\cdots \rho_{T|\mathcal{F}_{T-1}}(Z)\right) \tag{1.6}$$

of the constructed conditional risk mappings. Assuming that each conditional risk measure is coherent, we have that the composite risk measure ϱ is also coherent.

The composite risk measure ϱ assigns to each state of the process (node of the scenario tree) the corresponding conditional risk measure. Consequently the formulation (1.5) is time consistent with respect to optimization at every state performed by minimizing the corresponding conditional risk measure (cf. [7]). This raises a natural question whether for a given risk measure $\rho : \mathcal{Z} \to \mathbb{R}$, the optimization problem (1.3) is equivalent to the time consistent problem (1.5) for some choice of the conditional risk measures $\rho_{t|\mathcal{F}_{t-1}}$. Of course, a sufficient condition for such time consistency of problem (1.3) is that risk measure ρ can be represented as the composite risk measure ϱ, i.e., $\rho(\cdot) = \varrho(\cdot)$ for some choice of the conditional risk measures. We say that risk measure $\rho : \mathcal{Z} \to \mathbb{R}$ is *decomposable* into a sequence $\rho_{t+1|\mathcal{F}_t}$, $t = 2, \dots, T$, of conditional risk mappings if

$$\rho(\cdot) = \rho_{2|\mathcal{F}_1}\left(\cdots \rho_{T|\mathcal{F}_{T-1}}(\cdot)\right). \tag{1.7}$$

Note that with a law invariant risk measure is associated its conditional analogue. If the decomposition (1.7) holds for a law invariant risk measure ρ with the conditional risk mappings $\rho_{t+1|\mathcal{F}_t}$ being the conditional analogues of ρ, then ρ is called time consistent in the mathematical finance literature (e.g., [4]). An example of time consistent risk measure is the expectation operator, i.e., $\rho(\cdot) := \mathbb{E}(\cdot)$ and $\rho_{t|\mathcal{F}_{t-1}}(\cdot) = \mathbb{E}[\cdot|\mathcal{F}_{t-1}]$. Another example is the essential supremum operator. It is shown in [5] that (under certain regularity conditions) for the class of law invariant coherent risk measures these two examples are the only possible time consistent risk measures. In fact a more general result is proved in [5]; it is shown that in the class of law invariant convex risk measures, time consistent risk measures can be only entropic risk measures.

We approach the question of decomposability of a coherent risk measure from a somewhat different point of view. We show that if each conditional risk mapping in (1.6) is law invariant and coherent, then the corresponding composite risk measure can be law invariant only in the two cases of expectation and essential supremum operators. We will use a framework of finite scenario trees and the derivations will be rather elementary.

2. Main result

Let us consider the following construction. Let $\Omega = \{\omega_1, \dots, \omega_n\}$, $n \ge 3$, be a finite sample space equipped with equal probabilities $p_i = 1/n$, $i = 1, \dots, n$. Consider the sample space $\Omega \times \Omega$ equipped with equal probabilities $1/n^2$ and sigma algebra $\mathcal{F}$ of all its subsets. Denote $\mathcal{Z}$ the linear space of functions (random variables) $Z : \Omega \times \Omega \to \mathbb{R}$. We can identify random variable $Z \in \mathcal{Z}$ with the corresponding $n \times n$ matrix with entries $Z_{ij} := Z(\omega_i, \omega_j)$, $i, j = 1, \dots, n$. Denote by $Z_{i*} := (Z_{i1}, \dots, Z_{in})$ and $Z_{*i} := (Z_{1i}, \dots, Z_{ni})$ the respective rows and columns of matrix Z. Let $\mathcal{G}$ be the subalgebra of $\mathcal{F}$ generated by events $\{\omega_i\} \times \Omega$, $i = 1, \dots, n$, and $\mathcal{Z}_\mathcal{G}$ be the linear subspace of $\mathcal{Z}$ of $\mathcal{G}$-measurable functions, i.e., $Z \in \mathcal{Z}_\mathcal{G}$ if $Z_{i1} = \cdots = Z_{in}$ for every $i \in \{1, \dots, n\}$.

Let $\rho_0 : \mathcal{Z}_\mathcal{G} \to \mathbb{R}$ be a real valued law invariant coherent risk measure, and ρ_i, $i = 1, \dots, n$, be real valued law invariant coherent risk measures defined on the space of random variables $Y : \Omega \to \mathbb{R}$. We can view random variable $Y : \Omega \to \mathbb{R}$ as n-dimensional vector with entries $Y_j := Y(\omega_j)$, $j = 1, \dots, n$, and hence to consider values $\rho_i(Z_{i*})$ for $i \in \{1, \dots, n\}$. Also we can view risk measures ρ_i, $i = 0, 1, \dots, n$, as functions $\rho_i : \mathbb{R}^n \to \mathbb{R}$. Consider the risk mapping $\rho_{|\mathcal{G}} : \mathcal{Z} \to \mathcal{Z}_\mathcal{G}$ defined as

$$[\rho_{|\mathcal{G}}(Z)](\{\omega_i\} \times \Omega) = \rho_i(Z_{i*}), \quad i = 1, \dots, n, \tag{2.1}$$

and the composite risk measure $\varrho : \mathcal{Z} \to \mathbb{R}$ defined as

$$\varrho(Z) := \rho_0(\rho_{|\mathcal{G}}(Z)). \tag{2.2}$$

Note that law invariance of risk measures ρ_i, $i = 0, 1, \dots, n$, means that if $Y, Y' \in \mathbb{R}^n$ are such that components of Y' are obtained by a permutation of components of Y, then $\rho_i(Y) = \rho_i(Y')$. Similarly, ϱ is law invariant if $Z \in \mathcal{Z}$ and Z' is obtained from Z by a permutation of components of Z, then $\varrho(Z) = \varrho(Z')$.

Lemma 2.1. *Let ρ_i, $i = 0, 1, \dots, n$, be law invariant coherent risk measures and ϱ be the corresponding composite risk measure defined in (2.2). Suppose that ϱ is law invariant. Then $\rho_0 = \rho_1 = \cdots = \rho_n$.*

Proof. First, let us show that $\rho_1 = \cdots = \rho_n$. Indeed, suppose for example that $\rho_1 \ne \rho_2$. This means that there exists $Y \in \mathbb{R}^n$ such that $\rho_1(Y) \neq \rho_2(Y)$. We can assume that $\rho_1(Y) > \rho_2(Y) > 0$. Let $Z \in \mathcal{Z}$ be such that $Z_{1*} = Y$ and all other components of Z are zeros, and $Z' \in \mathcal{Z}$ be such that $Z'_{2*} = Y$ and all other components of Z' are zeros. Note that Z' can be obtained from Z by permutation of its components and hence by law invariance of ϱ it follows that $\varrho(Z) = \varrho(Z')$. We have that $\rho_{|\mathcal{G}}(Z)$ and $\rho_{|\mathcal{G}}(Z')$ are given by vectors $X := (\rho_1(Y), 0, \dots, 0) \in \mathbb{R}^n$ and $X' := (0, \rho_2(Y), \dots, 0)$, respectively. Since ρ_0 is law invariant we have that $\rho_0(X) = \rho_0(X')$. Also by the positive homogeneity of ρ_0 we have that $\rho_0(X) = \rho_1(Y)\rho_0(e_1)$ and $\rho_0(X') = \rho_2(Y)\rho_0(e_1)$, where $e_i \in \mathbb{R}^n$ denotes the i-th coordinate vector. Also

$$1 = \rho_0(e_1 + \cdots + e_n) \le \rho_0(e_1) + \cdots + \rho_0(e_n) = n\rho_0(e_1),$$

and hence $\rho_0(e_1) \ge 1/n > 0$. Thus $\varrho(Z) > \varrho(Z')$, a contradiction.

Let ρ denote the common risk measure ρ_i, $i = 1, \dots, n$. Next let us show that $\rho_0 = \rho$. Consider $Y \in \mathbb{R}^n$ and let $Z \in \mathcal{Z}$ be such that $Z_{i*} = Y$, $i = 1, \dots, n$. Then $\rho(Z_{i*}) = \rho(Y)$, $i = 1, \dots, n$, and hence $\rho_{|\mathcal{G}}(Z) = (\rho(Y), \dots, \rho(Y))$ and thus $\varrho(Z) = \rho(Y)$. On the other hand, let $Z' \in \mathcal{Z}$ be such that $Z_{*i} = Y$, $i = 1, \dots, n$, i.e., $Z' = Z^{\mathsf{T}}$. Note that Z' can be obtained from Z by permutation of its components and hence $\varrho(Z') = \varrho(Z)$. Moreover, $Z_{i*} = (Y_i, \dots, Y_i)$ and hence $\rho(Z_{i*}) = Y_i$. Thus $\rho_{|\mathcal{G}}(Z') = Y$ and $\varrho(Z') = \rho_0(Y)$. It follows that $\rho_0(Y) = \rho(Y)$. □

Remark 1. It could be noted that positive homogeneity of the considered risk measures was used in the above proof only to ensure that if $c_1 > c_2 > 0$, then $\rho_0(c_1 e_1) > \rho_0(c_2 e_1)$.

For a finite set Ω and $Y : \Omega \to \mathbb{R}$ denote

$$\rho^{\max}(Y) := \max_{\omega \in \Omega} Y(\omega) \tag{2.3}$$

the max-measure. Note that if $\rho_i := \rho^{\max}$, $i = 0, 1, \dots, n$, then the composite risk measure ϱ is also the max-measure.

Theorem 2.1. *Let ρ_i, $i = 0, 1, \dots, n$, be law invariant coherent risk measures and ϱ be the corresponding composite risk measure defined in (2.2). Suppose that ϱ is law invariant. Then $\rho_i = \rho_0$, $i = 1, \dots, n$, and ρ_0 is either the expectation $\rho_0(\cdot) = \mathbb{E}[\cdot]$ or the max-measure $\rho_0(\cdot) = \rho^{\max}(\cdot)$.*

Proof. By Lemma 2.1 we have that risk measures ρ_i, $i = 0, 1, \dots, n$, are equal to each other. Denote by ρ the common risk measure ρ_i. Since ρ is coherent it has the following dual representation (cf. [1])

$$\rho(Y) = \sup_{a \in \mathfrak{A}} a^{\mathsf{T}} Y, \tag{2.4}$$

where $\mathfrak{A}$ is a convex compact subset of $\Delta_n = \{a \in \mathbb{R}^n_+ : \sum_{i=1}^n a_i = 1\}$. It follows that for any $Z \in \mathcal{Z}$,

$$\rho_{|\mathcal{G}}(Z) = \left(\sup_{b_1 \in \mathfrak{A}} b_1^{\mathsf{T}} Z_{1*}, \dots, \sup_{b_n \in \mathfrak{A}} b_n^{\mathsf{T}} Z_{n*} \right), \tag{2.5}$$

and hence for any $Z \in \mathcal{Z}$ with nonnegative components $Z_{ij} \ge 0$,

$$\varrho(Z) = \sup_{a \in \mathfrak{A}} a^{\mathsf{T}} \rho_{\mathcal{G}}(Z) \ge \sup_{a \in \mathfrak{A}} \sum_{i,j=1}^n a_i a_j Z_{ij}. \tag{2.6}$$

Now consider $Y \in \mathbb{R}^n$ with positive components arranged in the decreasing order, i.e., $Y_1 > \cdots > Y_n > 0$, and $Z \in \mathcal{Z}$ such that $Z_{i*} = Y$, $i = 1, \dots, n$. Since the maximum in the right hand side of (2.4) is attained, there exists $\bar{a} \in \mathfrak{A}$ (depending on Y) such that $\rho(Y) = \bar{a}^{\mathsf{T}} Y$. Recall that $\bar{a} \ge 0$ and $\bar{a}_1 + \cdots + \bar{a}_n = 1$, and note that since components of Y are decreasing and ρ is law invariant, it follows that $\bar{a}_1 \ge \cdots \ge \bar{a}_n$. We have that $\rho_{\mathcal{G}}(Z) = (\rho(Y), \dots, \rho(Y))$ and hence $\bar{a}^{\mathsf{T}} \rho_{|\mathcal{G}}(Z) = \rho(Y)$, and thus

$$\varrho(Z) = \rho(Y) = \bar{a}^{\mathsf{T}} \rho_{|\mathcal{G}}(Z) = \sum_{i=1}^n \bar{a}_i \sum_{j=1}^n \bar{a}_j Y_j = \sum_{i,j=1}^n \bar{a}_i \bar{a}_j Y_j. \tag{2.7}$$

Since ϱ is law invariant, we have that if $Z' \in \mathcal{Z}$ is obtained from the matrix Z by a permutation of its elements, then $\varrho(Z) = \varrho(Z')$ and hence by (2.6) and (2.7) it follows that

$$\sum_{i,j=1}^n \bar{a}_i \bar{a}_j Y_j \ge \sum_{i,j=1}^n \bar{a}_i \bar{a}_j Z'_{ij}. \tag{2.8}$$

In particular, consider $Z' \in \mathcal{Z}$ obtained by interchanging elements $Z_{12} = Y_2$ and $Z_{i1} = Y_1$, $i \ge 3$, of Z. By (2.8) we obtain that

$$\bar{a}_1 \bar{a}_2 Y_2 + \bar{a}_i \bar{a}_1 Y_1 \ge \bar{a}_1 \bar{a}_2 Y_1 + \bar{a}_i \bar{a}_1 Y_2.$$

It follows that $\bar{a}_1 \bar{a}_i (Y_1 - Y_2) \ge \bar{a}_1 \bar{a}_2 (Y_1 - Y_2)$, which implies that $\bar{a}_i \ge \bar{a}_2$ and hence $\bar{a}_i = \bar{a}_2$, $i = 3, \dots, n$. That is, we obtain that $\bar{a}_2 = \cdots = \bar{a}_n$.

Now let $Z' \in \mathcal{Z}$ be obtained from Z by interchanging elements $Z_{22} = Y_2$ and $Z_{13} = Y_3$. Again by (2.8) we have that

$$\bar{a}_2^2 Y_2 + \bar{a}_1 \bar{a}_3 Y_3 \ge \bar{a}_2^2 Y_3 + \bar{a}_1 \bar{a}_3 Y_2,$$

and since $\bar{a}_2 = \bar{a}_3$ it follows that $\bar{a}_2^2 \ge \bar{a}_1 \bar{a}_2$. Consequently either $\bar{a}_2 = 0$ or $\bar{a}_1 = \bar{a}_2$. That is, either $\bar{a} = e_1$ or $\bar{a} = (e_1 + \cdots + e_n)/n$, i.e., either $\rho(Y) = \max_{1 \le i \le n} Y_i$ or $\rho(Y) = n^{-1}(Y_1 + \cdots + Y_n) = \mathbb{E}[Y]$. By continuity of $\rho(\cdot)$ this shows that either $\rho(\cdot) = \mathbb{E}[\cdot]$ or ρ is the max-measure on $\mathbb{R}^n_+$. Since $\rho(Y + c) = \rho(Y) + c$ for any $Y \in \mathbb{R}^n$ and $c \in \mathbb{R}$, this holds on $\mathbb{R}^n$ as well. □

Remark 2. It is essential in the above result that all probabilities p_i are the same. Otherwise suppose that the probabilities p_i are such that $\sum_{i \in I} p_i = \sum_{j \in J} p_j$, for $I, J \subset \{1, \dots, n\}$, only if $I = J$, and the same holds for probabilities $p_{ij} = p_i p_j$ on the space $\Omega \times \Omega$. Then any risk measures defined on the spaces of random variables $Y : \Omega \to \mathbb{R}$ and $Z : \Omega \times \Omega \to \mathbb{R}$ are law invariant.

Remark 3. The result of Theorem 2.1 has the following interpretation. Consider the following 3-stage scenario tree. The root node at the first stage has $n \ge 3$ children nodes and every node at the second stage has n children nodes. This results in n^2 scenarios, to each scenario we assign equal probability $1/n^2$, so that the probability of moving from one node to a node at the next stage is $1/n$. Suppose that the root node is assigned a law invariant coherent risk measure defined on the space of its children nodes, and every node at the second stage is assigned a law invariant coherent risk measure on its children nodes (these risk measures can be different from each other). This defines a composite risk measure ϱ on the considered scenario tree (cf. [9, section 6.7.1]). By Theorem 2.1 we have that if ϱ is law invariant, then ϱ is either the expectation or the max-risk measure. For example the AVaR_α, $\alpha \in (0, 1)$, cannot be represented as such a composite risk measure.

Remark 4. In a similar way this can be extended to the following continuous case. Let $\Xi_1 \subset \mathbb{R}^{d_1}$ and $\Xi_2 \subset \mathbb{R}^{d_2}$ be two closed sets equipped with respective probability distributions P_1 and P_2. Suppose that the probability measures P_1 and P_2 are atomless and complete and let $\Xi := \Xi_1 \times \Xi_2$ be equipped with the probability measure $P := P_1 \times P_2$ and the sigma algebra $\mathcal{F}$ of the corresponding Lebesgue sets (see, e.g., [2, p. 45]). Consider space $\mathcal{Z} := L_p(\Xi, \mathcal{F}, P)$, with $p \in [1, \infty)$, of random variables $Z : \Xi \to \mathbb{R}$ having finite p-th order moments. Conditional on ξ_1 define a (real valued) law invariant coherent risk measure $\rho_{|\xi_1}(\cdot)$ on the space of random variables $Y : \Xi_2 \to \mathbb{R}$ having finite p-th order moments. For a (real valued) law invariant coherent risk measure ρ_0, defined on the space of random variables $Y : \Xi_1 \to \mathbb{R}$ having finite p-th order moments, consider the corresponding composite risk measure $\varrho(\cdot) := \rho_0(\rho_{|\xi_1}(\cdot)) : \mathcal{Z} \to \mathbb{R}$. Then $\varrho(\cdot)$ can be law invariant only if $\varrho(\cdot) = \mathbb{E}[\cdot]$. Recall that we consider real valued risk measures and note that the essential supremum risk measure $\varrho(Z) = \operatorname{ess\,sup}(Z)$ can have ∞ values for some $Z \in L_p(\Xi, \mathcal{F}, P)$, with $p \in [1, \infty)$. If we consider the space $\mathcal{Z} := L_\infty(\Xi, \mathcal{F}, P)$ of essentially bounded random variables, then the composite risk measure can be law invariant if either $\varrho(\cdot) = \mathbb{E}[\cdot]$ or $\varrho(\cdot) = \operatorname{ess\,sup}(\cdot)$.

3. Concluding remarks

We showed, in the framework of finite scenario trees, that if each conditional risk mapping is law invariant and coherent, then the corresponding composite risk measure can be law invariant only in the two cases of the expectation and max-risk measures. This implies, for example, that $\rho(\cdot) := \mathsf{AVaR}_\alpha(\cdot)$, $\alpha \in (0, 1)$, risk measure is not representable as a composition of a sequence of law invariant coherent risk mappings.

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