Qiang Huang
e-mail: huangq@eng.usf.edu
Department of Industrial and Management Systems Engineering,
University of South Florida,
Tampa, Florida 33620

Jianjun Shi
Department of Industrial and Operations Engineering,
The University of Michigan,
Ann Arbor, MI 48109

Stream of Variation Modeling and Analysis of Serial-Parallel Multistage Manufacturing Systems

In a Serial-Parallel Multistage Manufacturing System (SP-MMS), identical work-stations are utilized at each stage to meet the productivity and line balance requirements. In such a system, parts could go through different process routes and some routes may merge at certain stage(s). Due to the existence of multiple variation streams, it is challenging to model and analyze variation propagation in a system. This paper extends the state space modeling approach from single process route to the SP-MMS with multiple routes. Several model dimension reduction techniques are proposed to reduce model complexity. Properties of these techniques are studied from the perspectives of system representation and diagnosability. Furthermore, these techniques are applied to analyze system measurement strategies. [DOI: 10.1115/1.1765149]

1 Introduction

A multistage manufacturing system (MMS) involves multiple stages or operations to fabricate a product. Examples of MMSs include engine head machining systems or automotive body assembly systems. To meet productivity and line balance requirements, a MMS usually utilizes identical work-stations at each stage. This type of systems is called Serial-Parallel MMS (SP-MMS). The impact of MMS configuration on system performance has been studied by Koren et al. [1]. In such a system, each part follows the same processing sequence, i.e., sequentially going through every stage once. However, process routes may vary from part to part. For instance, in a three-stage serial-parallel machining system illustrated in Fig. 1, a part could go through process route 1, i.e., through the machine tool No. 1 at each stage; or through other routes. ($\mathbf{x}_0$ denotes raw workpiece deviation and $\mathbf{x}_k^{(i)}$ denotes the part deviation after stage k through process route i.) As an example, only portion of all potential routes is shown in the figure. Since part variation streams from different routes are not necessary to be the same, the challenging issues are how to model and analyze the multiple variation streams in a SP-MMS.

In the field of Statistical Process Control, control charts have been developed to monitor multiple stream processes [2,3]. It mainly focuses on process change detection, as opposed to root cause identification. Recently, researches have been conducted to model and diagnose single variation stream problem in a MMS. Jin and Shi [4] developed state space model to depict variation propagation in assembly processes. By developing a state transition model, Mantripragada and Whitney [5] modeled the entire assembly sequence as a set of discrete events to simulate and predict the propagation of variation in mechanical assemblies. Lawless et al. [6] and Agrawal et al. [7] investigated variation transmission in both assembly and machining process by using an AR(1) model. State space modeling approach was further extended to model multistage machining processes [8–10]. Root cause identification has also been studied for single variation stream in assembly processes [11] and machining processes [12]. If no two process routes merge at certain stage(s), the previous work can be directly applied to a SP-MMS by studying every process route separately. If two process routes share at least one work-station, i.e., merge at one stage, there is a need to extend previous methodologies by considering all routes and their interactions. As such, global optimal solutions are expected for SP-MMS.

Gauging/sensing strategy is a good example to illustrate the necessities of extending the existing methodologies to SP-MMS. In Fig. 1, parts would be measured from all six routes to identify the root causes if the routes are studied separately. Intuitively it might be sufficient to take measurements, e.g., only from process routes 1, 3, and 6, because all eight machines in the systems are involved in those three routes and root causes might be identified with given measurements. Systematic approach is preferred not only for gauging strategy, but also for closely related system monitoring and root cause identification problems. Previous methodologies need to be extended to the case of multiple variation streams because SP-MMS has been adopted as a common configuration in industries [1].

The focus of this paper is to develop a generic system-level methodology to model and analyze multiple variation streams in a SP-MMS. Section 2 extends the state space modeling approach to the SP-MMS. Section 3 discusses the model dimension issues and proposes **u** and **y** reduction techniques to reduce model dimensions. The impacts of **u** and **y** reduction techniques on system representation and system diagnosability are studied in Section 4. Section 5 analyzes different measurement strategies based on system model and **u** and **y** reduction techniques. The conclusion is given in Section 6.

2 Variation Modeling of SP-MMSs with Multiple Process Routes

2.1 Modeling of System Variation Streams. Assume the total number of process routes is R in an N-stage SP-MMS. Deviation of part features are represented as a vector $\mathbf{x}$ by using vectorial surface model [8,13]. For example, the ith part feature $\mathbf{S}_i$ in Fig. 2 can be modeled by a normal vector to $\mathbf{S}_i$, i.e., (n_{xi}, n_{yi}, n_{zi}), a point on $\mathbf{S}_i$, i.e., (p_{xi}, p_{yi}, p_{zi}), and size, e.g., diameter of a cylindrical surface. Let $\mathbf{x}_0$ represent raw workpiece deviation. Denote by $\mathbf{x}_k^{(i)}$ the part deviation after stage k through process route i (See the example in Fig. 1). Superscript "$^{(i)}$" denotes process route i $(i=1,2,\ldots,R)$ and subscript "k" denotes stage k $(k=1,2,\ldots,N)$. The conventions will be followed hereafter. A vector $\mathbf{y}_k^{(i)}$ denotes deviation of quality characteristics generated after stage k. Note that measurements are not necessary to be taken at each stage of process route i. By treating part deviation

Contributed by the Manufacturing Engineering Division for publication in the JOURNAL OF MANUFACTURING SCIENCE AND ENGINEERING. Manuscript received June 2003; Revised Feb. 2004. Associate Editor: S. Raman.

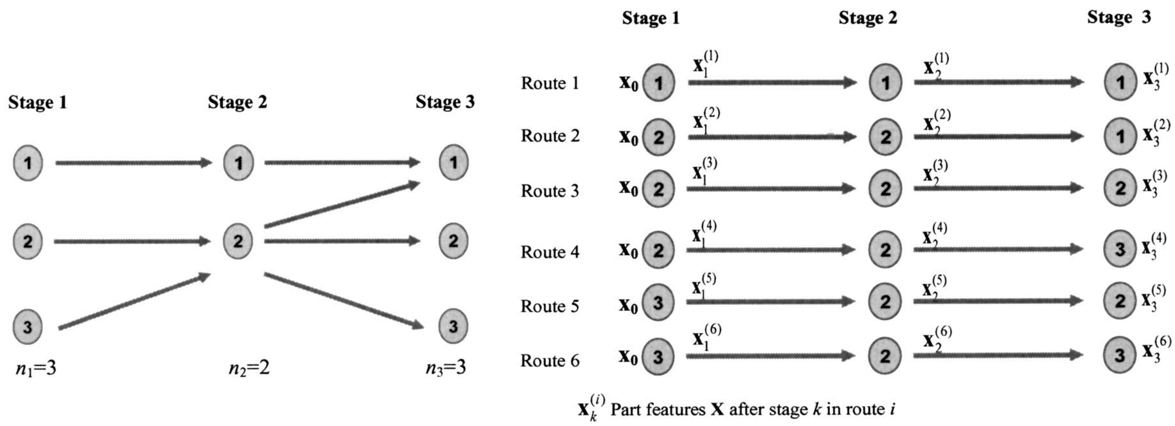


Fig. 1 Process route in a SP-MMS

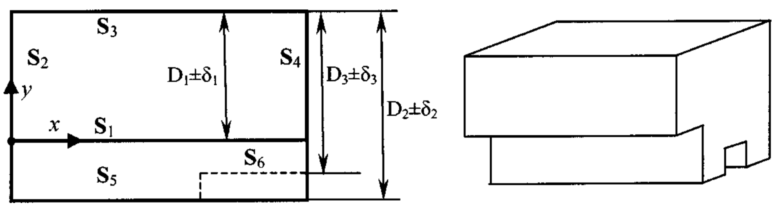


Fig. 2 Block part

$\mathbf{x}_k^{(i)}$ as a state vector and stage index as time index, the state space modeling approach can be applied to model the variation propagation for every single route i:

$$\mathbf{x}_k^{(i)}=\mathbf{A}_{k-1}^{(i)}\mathbf{x}_{k-1}^{(i)}+\mathbf{B}_k^{(i)}\mathbf{u}_k^{(i)}+\boldsymbol{\xi}_k^{(i)}, \quad k=1,\ldots,N; i=1,\ldots,R \qquad (1)$$

$$\mathbf{y}_k^{(i)}=\mathbf{C}_k^{(i)}\mathbf{x}_k^{(i)}+\boldsymbol{\eta}_k^{(i)}, \quad \{k\}\subset\{1,\ldots,N\}. \qquad (2)$$

where input vector $\mathbf{u}_k^{(i)}$ represents process deviations from the fixture and the machine tool at stage k of route i. $\mathbf{u}_k^{(i)}$'s have the same dimension for all i ($i=1,2,\ldots,R$) at stage k. Input matrix $\mathbf{B}_k^{(i)}$ and state transition matrix $\mathbf{A}_{k-1}^{(i)}$ transfer process deviations and incoming workpiece deviation to state vector $\mathbf{x}_k^{(i)}$, respectively. The physics underlying $\mathbf{A}_{k-1}^{(i)}$ and $\mathbf{B}_k^{(i)}$ transformations involves fixturing and cutting operation at stage k. Since operations are modeled as kinematic transformations in this study, $\mathbf{A}_{k-1}^{(i)}$ and $\mathbf{B}_k^{(i)}$ are constant matrices determined only by product and process design. Matrix $\mathbf{C}_k^{(i)}$ maps part deviation $\mathbf{x}_k^{(i)}$ to $\mathbf{y}_k^{(i)}$, which characterizes the geometric relationship between product characteristics and part features. $\boldsymbol{\xi}_k^{(i)}$ and $\boldsymbol{\eta}_k^{(i)}$ are error terms. The detailed process-level model derivation can be referred to [4] for assembly processes and [8–10] for machining processes.

Following assumptions are made to model multiple variation streams.

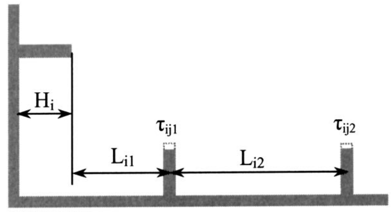


Fig. 3 Fixture locating scheme and locator deviations

A1. All process routes use the same batch of workpiece. In another word, raw workpiece deviation $\mathbf{x}_0^{(i)}$'s follow the same distribution as $\mathbf{x}_0$, where $\mathbf{x}_0$ is negligible if workpiece is of high quality.

A2. The machine tools and operations at the same stage are identical. Different process routes are expected to perform the same fixturing and cutting operations at stage k. Therefore, the system matrices $[\mathbf{A}_{k-1}^{(i)}\mathbf{B}_k^{(i)}\mathbf{C}_k^{(i)}]$ by design are the same for all i. Superscript will be dropped hereafter.

A3. If routes i and j merge at stage k, the input random vectors $\mathbf{u}_k$'s for those two routes at that stage are assumed to be the same, i.e., $\mathbf{u}_k^{(i)}=\mathbf{u}_k^{(j)}$.

A4. The error terms $[\boldsymbol{\xi}_k^{(i)}\boldsymbol{\eta}_k^{(i)}]$, which represent the normal production conditions within designated tooling tolerance, are assumed to be the same for every route. The superscript will be dropped too.

A remark is given as follows:

R1 These four assumptions are made by considering the fact in a real engine machining plant. By design, all process routes should be identical and well-maintained. Significant deviation from design needs to be detected through monitoring mechanism (not discussed in this paper). Assuming distributions for error terms $\boldsymbol{\xi}_k^{(i)}$ and $\boldsymbol{\eta}_k^{(i)}$ in A4 is more critical for process condition monitoring than for the topic of model dimension reduction investigated in this paper. When distributions are necessary, assumptions need to be investigated based on a given product and manufacturing process. For instance, if tool wear is a concern in production, then a Gaussian random process with correlation among stages is more reasonable than the assumption of normal distributions with independent identically distributed property.

Let $\mathbf{u}_\cdot^{(i)}=[\mathbf{u}_1^{(i)^T},\mathbf{u}_2^{(i)^T},\ldots,\mathbf{u}_N^{(i)^T}]^T$ be the process deviations from operations 1 to N of route i and $\mathbf{y}_\cdot^{(i)}=[\mathbf{y}_1^{(i)^T},\mathbf{y}_2^{(i)^T},\ldots,\mathbf{y}_N^{(i)^T}]^T$ be the deviations of all measured characteristics in route i. Note that measurement might not be taken at each stage. By following the procedure in [14],

$$\mathbf{y}_{\bullet}^{(i)} = \mathbf{\Gamma}_{\bullet}\mathbf{u}_{\bullet}^{(i)} + \mathbf{\Gamma}_{\bullet 0}\mathbf{x}_0 + \boldsymbol{\varepsilon}_{\bullet}, \quad \text{where } \mathbf{\Gamma}_{\bullet} = \begin{bmatrix} \mathbf{C}_1\mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,1}\mathbf{B}_1 & \mathbf{C}_2\mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,1}\mathbf{B}_1 & \mathbf{C}_N\mathbf{\Phi}_{N,2}\mathbf{B}_2 & \cdots & \mathbf{C}_N\mathbf{B}_N \end{bmatrix}, \quad \mathbf{\Gamma}_{0\bullet} = \begin{bmatrix} \mathbf{C}_1\mathbf{\Phi}_{1,0} \\ \mathbf{C}_2\mathbf{\Phi}_{2,0} \\ \vdots \\ \mathbf{C}_N\mathbf{\Phi}_{N,0} \end{bmatrix},$$

$$\mathbf{\Phi}_{k,j} = \begin{cases} \mathbf{A}_{k-1}\mathbf{A}_{k-2}\cdots\mathbf{A}_j, & k \geq j+1 \\ \mathbf{I}, & k = j \end{cases}, \quad \boldsymbol{\varepsilon}_{\bullet} = [\boldsymbol{\varepsilon}_1^T, \boldsymbol{\varepsilon}_2^T, \cdots, \boldsymbol{\varepsilon}_N^T]^T, \quad \text{and} \quad \boldsymbol{\varepsilon}_k = \Sigma_{n=1,k}\mathbf{C}_k\mathbf{\Phi}_{k,n}\boldsymbol{\xi}_n + \boldsymbol{\eta}_k.$$

Generally, the observed part deviations in a SP-MMS with R routes can be modeled as:

$$\mathbf{y} = \mathbf{\Gamma}\mathbf{u} + \mathbf{\Gamma}_0\mathbf{x}_0 + \boldsymbol{\varepsilon}, \tag{3}$$

where $\mathbf{y} = [\mathbf{y}_{\bullet}^{(1)^T}, \mathbf{y}_{\bullet}^{(2)^T}, \ldots, \mathbf{y}_{\bullet}^{(R)^T}]^T$, $\mathbf{\Gamma} = \text{diag}(\mathbf{\Gamma}_{\bullet}, \mathbf{\Gamma}_{\bullet}, \ldots, \mathbf{\Gamma}_{\bullet})$, $\mathbf{u} = [\mathbf{u}_{\bullet}^{(1)^T}, \mathbf{u}_{\bullet}^{(2)^T}, \ldots, \mathbf{u}_{\bullet}^{(R)^T}]^T$, $\mathbf{\Gamma}_0 = [\mathbf{\Gamma}_{0\bullet}^T, \mathbf{\Gamma}_{0\bullet}^T, \cdots, \mathbf{\Gamma}_{0\bullet}^T]^T$, and $\boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_{\bullet}^T, \boldsymbol{\varepsilon}_{\bullet}^T, \cdots, \boldsymbol{\varepsilon}_{\bullet}^T]^T$.

2.2 Block Part Example. A machining system of fabricating block parts is given to illustrate the system model (3). The part is composed of six surfaces: $\mathbf{S}_1 - \mathbf{S}_6$ (Fig. 2). To meet the specifications for dimensions $D_1 - D_3$, three operations are selected. The first operation is to use datum surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ to mill $\mathbf{S}_3$. In operation 2, $\mathbf{S}_2$ and $\mathbf{S}_3$ are chosen as datums to mill $\mathbf{S}_5$. The last operation is to mill a slot $\mathbf{S}_6$ with the same datums used in the second operation. These operations are performed in the machining system depicted by Fig. 1.

Let τ_{ijk} denote the deviation of the kth ($k=1,2,\ldots,6$) locator in fixture j ($j=1,2,\ldots,n_i$) at operation i ($i=1,2,3$). Assume fixture j is mounted on machine tool j at that operation. Figure 3 illustrates the fixture locating scheme, which is specified by geometric dimensions H_i, L_{i1}, and L_{i2} for operation i.

Denote by γ_{ij} and δ_{ij} the angular and positional deviations of machine tool j at operation i. The process deviations of route 1 are $\mathbf{u}_1^{(1)} = [\tau_{111}, \tau_{112}, \gamma_{11}, \delta_{11}]^T$, $\mathbf{u}_2^{(1)} = [\tau_{211}, \tau_{212}, \delta_{21}, \gamma_{21}]^T$, and $\mathbf{u}_3^{(1)} = [\tau_{311}, \tau_{312}, \delta_{31}, \gamma_{31}]^T$ for the three operations. For process route 1, let $\mathbf{x}_0 = \mathbf{0}^T$, $\mathbf{x}_1^{(1)} = (\Delta n_{x3}, \Delta p_{y3}, 0,0,0,0)^T$, $\mathbf{x}_2^{(1)} = (\Delta n_{x3}, \Delta p_{y3}, \Delta n_{x5}, \Delta p_{y5}, 0,0)^T$, $\mathbf{x}_3^{(1)} = (\Delta n_{x3}, \Delta p_{y3}, \Delta n_{x5}, \Delta p_{y5}, \Delta n_{x6}, \Delta p_{y6})^T$, $\mathbf{u}_{\bullet}^{(1)} = [\tau_{111}, \tau_{112}, \gamma_{11}, \delta_{11}, \tau_{211}, \tau_{212}, \delta_{21}, \gamma_{21}, \tau_{311}, \tau_{312}, \delta_{31}, \gamma_{31}]^T$, $\mathbf{y}_1^{(1)} = \Delta D_1$, $\mathbf{y}_2^{(1)} = \Delta D_2$, and $\mathbf{y}_3^{(1)} = \Delta D_3$. Then

$$\mathbf{x}_3^{(1)} = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix} \mathbf{x}_2^{(1)} + \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ \frac{1}{L_{32}} & -\frac{1}{L_{32}} & 1 & 0 \\ \left(1+\frac{L_{31}}{L_{32}}\right) & \frac{L_{31}}{L_{32}} & -H_3 & -1 \end{pmatrix} \mathbf{u}_3^{(1)} + \boldsymbol{\xi}_3^{(1)}$$

$$= \begin{pmatrix} -\frac{1}{L_{12}} & \frac{1}{L_{12}} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\left(1+\frac{L_{11}}{L_{12}}\right) & \frac{L_{11}}{L_{12}} & H_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{L_{12}} & \frac{1}{L_{12}} & -1 & 0 & \frac{1}{L_{22}} & -\frac{1}{L_{22}} & 1 & 0 & 0 & 0 & 0 & 0 \\ -\left(1+\frac{L_{11}}{L_{12}}\right) & \frac{L_{11}}{L_{12}} & H_1 & 1 & \left(1+\frac{L_{21}}{L_{22}}\right) & \frac{L_{21}}{L_{22}} & -H_2 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{L_{12}} & \frac{1}{L_{12}} & -1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{L_{32}} & -\frac{1}{L_{32}} & 1 & 0 \\ -\left(1+\frac{L_{11}}{L_{12}}\right) & \frac{L_{11}}{L_{12}} & H_1 & 1 & 0 & 0 & 0 & 0 & \left(1+\frac{L_{31}}{L_{32}}\right) & \frac{L_{31}}{L_{32}} & -H_3 & -1 \end{pmatrix} \mathbf{u}_{\bullet}^{(1)} + \boldsymbol{\xi}_3^{(1)} \tag{4}$$

$$\mathbf{y}_{\bullet}^{(1)} = \begin{pmatrix} \Delta D_1 \\ \Delta D_2 \\ \Delta D_3 \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&0 \\ 0&1&0&-1&0&0 \\ 0&1&0&0&0&-1 \end{pmatrix} \mathbf{x}_3^{(1)} + \boldsymbol{\varepsilon}_{\bullet}^{(1)} = \mathbf{\Gamma}_{\bullet}\mathbf{u}_{\bullet}^{(1)} + \boldsymbol{\varepsilon}_{\bullet}^{(1)}, \tag{5}$$

where

$$\boldsymbol{\Gamma}_{\bullet}=\begin{pmatrix} -\left(1+\frac{L_{11}}{L_{12}}\right) & \frac{L_{11}}{L_{12}} & H_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(1+\frac{L_{21}}{L_{22}}\right) & \frac{L_{21}}{L_{22}} & -H_2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(1+\frac{L_{31}}{L_{32}}\right) & \frac{L_{31}}{L_{32}} & -H_3 & -1 \end{pmatrix}. \quad (6)$$

Equations (4)–(6), whose derivation can be referred to Huang and Shi [15], only depict the variation propagation of individual process route. To characterize the six process routes in the system, define $\mathbf{u}_{\bullet}^{(1)}=[\mathbf{u}_1^{(1)^T},\mathbf{u}_2^{(1)^T},\mathbf{u}_3^{(1)^T}]^T$, $\mathbf{u}_{\bullet}^{(2)}=[\mathbf{u}_1^{(2)^T},\mathbf{u}_2^{(2)^T},\mathbf{u}_3^{(2)^T}]^T,\ldots,\mathbf{u}_{\bullet}^{(6)}=[\mathbf{u}_1^{(6)^T},\mathbf{u}_2^{(6)^T},\mathbf{u}_3^{(6)^T}]^T$, and the corresponding $\mathbf{u}_{72\times 1}$, $\mathbf{y}_{18\times 1}$, and $\boldsymbol{\Gamma}_{18\times 72}$. As such, system model (3) can be obtained to model the six variation streams.

3 Reducing Model Dimensions through u and y Reductions

One of the major concerns about the system model (3) is its high dimension. If n_k denotes the number of machine tools at stage k, theoretically the total possible number of routes R equals to $\Pi_{k=1}^{N} n_k$. Thus system dimension increases dramatically with R. However, when some process routes merge together, system model (3) can be revised and the dimension might be greatly reduced. Before discussing the approaches of model reduction, we first classified the ways in which process routes merge.

There are three basic ways that two process routes merge together, i.e., coincidence, divergence, and convergence (Table 1). Per these three basic ways of route merging, Section 3.1 proposes $\mathbf{u}$ reduction technique to reduce the dimension of input vector $\mathbf{u}$.

Special combinations of the three basic ways of route merging, together with process information, make it possible to reduce not only $\mathbf{u}$ dimension, but also $\mathbf{y}$ dimension. Table 2 lists two special cases, where in the first case, two process routes coincide with each other until Stage M and diverge after then. Section 3.2 proposes $\mathbf{y}$ reduction technique to reduce not only the dimension of $\mathbf{u}$, but also the dimension of $\mathbf{y}$. In the second case of Table 2, two process routes diverge at Stage k and converge at Stage $k+\mathbf{S}$. Besides, the features machined at stage k will be used as datum in the process segment composed by stages $k+1$ to $k+\mathbf{S}$. Based on the process knowledge, Section 3.3 discusses reducing the dimensions of $\mathbf{u}$ and $\mathbf{y}$.

3.1 u Reduction or Γ Column Reduction. If routes i and j $(i<j)$ merge at stage k, then $\mathbf{u}_k^{(i)}=\mathbf{u}_k^{(j)}$ (by A3). Those two identical vectors, i.e., $\mathbf{u}_k^{(i)}$ and $\mathbf{u}_k^{(j)}$, can be merged into one sub-vector in $\mathbf{u}$ of model (3), i.e., only keeping $\mathbf{u}_k^{(i)}$. To describe the dimension reduction of $\boldsymbol{\Gamma}$, let $(\boldsymbol{\Gamma})_k^{(i)}$ be the block matrix in $\boldsymbol{\Gamma}$corresponding to $\mathbf{u}_k^{(i)}$. Note that $(\boldsymbol{\Gamma})_k^{(i)}$ has the same number of rows as $\boldsymbol{\Gamma}$. The reduced $\boldsymbol{\Gamma}$, denoted by $\boldsymbol{\Gamma}^{\mathbf{u}}$, is obtained by replacing $(\boldsymbol{\Gamma})_k^{(i)}$ with $(\boldsymbol{\Gamma})_k^{(i)}+(\boldsymbol{\Gamma})_k^{(j)}$ and deleting $(\boldsymbol{\Gamma})_k^{(j)}$ in original $\boldsymbol{\Gamma}$.

Example: The process routes 2 and 3 in Fig. 1, for instance, share machine tools at stages 1 and 2, i.e., $\mathbf{u}_1^{(2)}=\mathbf{u}_1^{(3)}$ and $\mathbf{u}_2^{(2)}=\mathbf{u}_2^{(3)}$. Before $\mathbf{u}$ reduction,

Table 1 Two routes merge at one stage

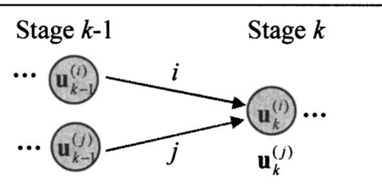
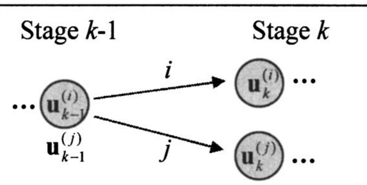
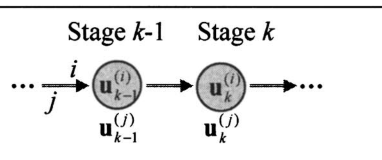
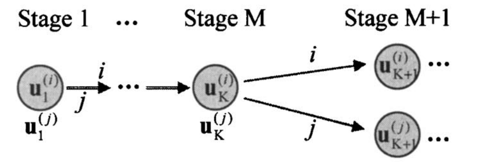
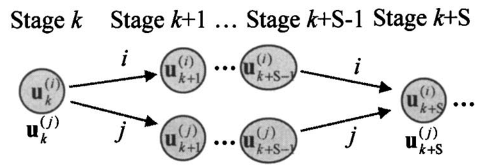
Classification	Description	Result
	1. Convergence: Routes i and j are merged at stage k.	$\mathbf{u}_{k-1}^{(i)} \neq \mathbf{u}_{k-1}^{(j)}$ $\mathbf{u}_k^{(i)} = \mathbf{u}_k^{(j)}$
	2. Divergence: Routes i and j are split at stage k.	$\mathbf{u}_{k-1}^{(i)} = \mathbf{u}_{k-1}^{(j)}$ $\mathbf{u}_k^{(i)} \neq \mathbf{u}_k^{(j)}$
	3. Coincidence: Routes i and j remained the same at stage k.	$\mathbf{u}_{k-1}^{(i)} = \mathbf{u}_{k-1}^{(j)}$ $\mathbf{u}_k^{(i)} = \mathbf{u}_k^{(j)}$

Table 2 Special combinations of route merging

Merging of Two Process Routes	Description	Result
	1. Routes i and j merge together from stage 1 to stage M, i.e., $\mathbf{u}_k^{(i)}=\mathbf{u}_k^{(j)}$ for k=1,2,…,M.	$\mathbf{y}_k^{(i)}=\mathbf{y}_k^{(j)}$, k=1,2, …,M
	2. Features machined at stage k used as datum in the process segment consisted of stages k+1 to k+S.	$\mathbf{y}_{k+S}^{(i)}=\mathbf{y}_{k+S}^{(j)}$

$$[\mathbf{y}_{\cdot}^{(2)^T},\mathbf{y}_{\cdot}^{(3)^T}]^T_{6\times 1} = \mathrm{diag}(\boldsymbol{\Gamma}_{\cdot},\boldsymbol{\Gamma}_{\cdot})_{6\times 24}(\mathbf{u}_1^{(2)^T},\mathbf{u}_2^{(2)^T},\mathbf{u}_3^{(2)^T},\mathbf{u}_1^{(3)^T},\mathbf{u}_2^{(3)^T},\mathbf{u}_3^{(3)^T})^T_{24\times 1}+\boldsymbol{\varepsilon}_0 \tag{7}$$

By conducting $\mathbf{u}$ reduction, $(\mathbf{u}_1^{(2)^T},\mathbf{u}_2^{(2)^T},\mathbf{u}_3^{(2)^T},\mathbf{u}_1^{(3)^T},\mathbf{u}_2^{(3)^T},\mathbf{u}_3^{(3)^T})^T_{24\times 1}$ is reduced to $(\mathbf{u}_1^{(2)^T},\mathbf{u}_2^{(2)^T},\mathbf{u}_3^{(2)^T},\mathbf{u}_3^{(3)^T})^T_{16\times 1}$. Correspondingly, $\mathrm{diag}(\boldsymbol{\Gamma}_{\cdot},\boldsymbol{\Gamma}_{\cdot})_{6\times 24}$ is reduced to $\boldsymbol{\Gamma}^{\mathbf{u}}_{6\times 16}$, where $\boldsymbol{\Gamma}^{\mathbf{u}}_{6\times 16}$ is

$$\boldsymbol{\Gamma}^{\mathbf{u}}=\begin{pmatrix}
-\left(1+\frac{L_{11}}{L_{12}}\right) & \frac{L_{11}}{L_{12}} & H_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \left(1+\frac{L_{21}}{L_{22}}\right) & \frac{L_{21}}{L_{22}} & -H_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(1+\frac{L_{31}}{L_{32}}\right) & \frac{L_{31}}{L_{32}} & -H_3 & -1 & 0 & 0 & 0 & 0\\
-\left(1+\frac{L_{11}}{L_{12}}\right) & \frac{L_{11}}{L_{12}} & H_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \left(1+\frac{L_{21}}{L_{22}}\right) & \frac{L_{21}}{L_{22}} & -H_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(1+\frac{L_{31}}{L_{32}}\right) & \frac{L_{31}}{L_{32}} & -H_3 & -1
\end{pmatrix}. \tag{8}$$

Model (7) is thus simplified as

$$[\mathbf{y}_{\cdot}^{(2)^T},\mathbf{y}_{\cdot}^{(3)^T}]^T_{6\times 1}=\boldsymbol{\Gamma}^{\mathbf{u}}_{6\times 16}(\mathbf{u}_1^{(2)^T},\mathbf{u}_2^{(2)^T},\mathbf{u}_3^{(2)^T},\mathbf{u}_3^{(3)^T})^T_{16\times 1}+\boldsymbol{\varepsilon}_0 \tag{9}$$

Here are some remarks:

R2 The advantage of $\mathbf{u}$ reduction is not just reducing the dimensions of system matrices. It is also necessary when the objective of study is to estimate $\mathbf{u}$, i.e., identifying root causes. $\mathbf{u}$ reduction could *increase the accuracy of estimating* $\mathbf{u}$ by pooling measurement data together for two identical variables. Since $\mathbf{u}$ reduction leads to reducing column size of $\boldsymbol{\Gamma}$, it is also called $\boldsymbol{\Gamma}$ column reduction.

R3 Note that when $\mathbf{u}_k^{(i)}=\mathbf{u}_k^{(j)}$, $\mathbf{y}_k^{(i)}=\mathbf{y}_k^{(j)}$ is unnecessarily true due to the possibility of $\mathbf{x}_{k-1}^{(i)}\neq\mathbf{x}_{k-1}^{(j)}$, i.e., the incoming workpieces might come from different process routes.

3.2 y Reduction or Γ Row Reduction. If routes i and j ($i<j$) merge together through stage M, i.e., $\mathbf{u}_k^{(i)}=\mathbf{u}_k^{(j)}$ for $k=1,2,\ldots,\mathrm{M}$, then $\mathbf{y}_k^{(i)}$ and $\mathbf{y}_k^{(j)}$ are identical random variables for $k=1,2,\ldots,\mathrm{M}$ (the first case in Table 2). In addition to $\mathbf{u}$ reduction, $\mathbf{y}$ can be reduced by eliminating all $\mathbf{y}_k^{(j)}$'s. The dimension of $\boldsymbol{\Gamma}$ is reduced by eliminating the corresponding rows.

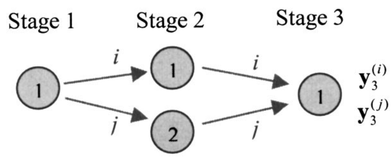


Fig. 4 y Reduction due to common datum

Table 3 u and y reduction for block part machining system

$\mathbf{u}$ reduction	$\mathbf{y}$ reduction
$\mathbf{u}_3^{(1)}=\mathbf{u}_3^{(2)}$	
$\mathbf{u}_1^{(2)}=\mathbf{u}_1^{(3)}=\mathbf{u}_1^{(4)}$ and $\mathbf{u}_2^{(2)}=\mathbf{u}_2^{(3)}=\mathbf{u}_2^{(4)}$	$\mathbf{y}_1^{(2)}=\mathbf{y}_1^{(3)}=\mathbf{y}_1^{(4)}$, $\mathbf{y}_2^{(2)}=\mathbf{y}_2^{(3)}=\mathbf{y}_2^{(4)}$
$\mathbf{u}_1^{(5)}=\mathbf{u}_1^{(6)}$ and $\mathbf{u}_2^{(2)}=\mathbf{u}_2^{(5)}=\mathbf{u}_2^{(6)}$	$\mathbf{y}_1^{(5)}=\mathbf{y}_1^{(6)}$, $\mathbf{y}_2^{(5)}=\mathbf{y}_2^{(6)}$
$\mathbf{u}_3^{(3)}=\mathbf{u}_3^{(5)}$	
$\mathbf{u}_3^{(4)}=\mathbf{u}_3^{(6)}$	

Since variables in $\mathbf{y}$ represent quality characteristics to be measured, reducing identical random variables in $\mathbf{y}$ implies *reduction of measurements*. Measurement reduction is very critical for a system with multiple variation streams because of the need of reducing gauging cost.

Example: For the same example illustrate in $\mathbf{u}$ reduction, $\mathbf{y}_1^{(2)}=\mathbf{y}_1^{(3)}$ and $\mathbf{y}_2^{(2)}=\mathbf{y}_2^{(3)}$ hold because of $\mathbf{u}_1^{(2)}=\mathbf{u}_1^{(3)}$ and $\mathbf{u}_2^{(2)}=\mathbf{u}_2^{(3)}$. As a result, $[\mathbf{y}_\cdot^{(2)^T},\mathbf{y}_\cdot^{(3)^T}]_{6\times 1}^T$ in (9) can be reduced to $[\mathbf{y}_\cdot^{(2)^T},\mathbf{y}_\cdot^{(3)^T}]_{4\times 1}^T$ by deleting $\mathbf{y}_1^{(3)}$ and $\mathbf{y}_2^{(3)}$. By conducting $\mathbf{y}$ reduction, model (9) is refined as

$$[\mathbf{y}_\cdot^{(2)^T},\mathbf{y}_\cdot^{(3)^T}]_{4\times 1}^T=\boldsymbol{\Gamma}_{4\times 16}^{\mathbf{y}}(\mathbf{u}_1^{(2)^T},\mathbf{u}_2^{(2)^T},\mathbf{u}_3^{(2)^T},\mathbf{u}_3^{(3)^T})_{16\times 1}^T+\boldsymbol{\varepsilon}_1 \tag{10}$$

where $\boldsymbol{\Gamma}_{4\times 16}^{\mathbf{y}}$ is

$$\boldsymbol{\Gamma}^{\mathbf{y}}=\begin{pmatrix} -\left(1+\frac{L_{11}}{L_{12}}\right) & \frac{L_{11}}{L_{12}} & H_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(1+\frac{L_{21}}{L_{22}}\right) & \frac{L_{21}}{L_{22}} & -H_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(1+\frac{L_{31}}{L_{32}}\right) & \frac{L_{31}}{L_{32}} & -H_3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(1+\frac{L_{31}}{L_{32}}\right) & \frac{L_{31}}{L_{32}} & -H_3 & -1 \end{pmatrix}. \tag{11}$$

3.3 y Reduction Due to Common Datum in a Process Segment. In machining systems, there is another opportunity to perform $\mathbf{y}$ reduction. Suppose the features machined at stage k will be used as datum in the process segment composed by stages $k+1$ to $k+\mathrm{S}$ (The second case in Table 2). Since $\mathbf{u}_k^{(i)}=\mathbf{u}_k^{(j)}$, $\mathbf{u}_{k+S}^{(i)}=\mathbf{u}_{k+S}^{(j)}$, and there is no datum change, $\mathbf{y}_{k+S}^{(i)}$ and $\mathbf{y}_{k+S}^{(j)}$ are identical random vectors. The dimensions of $\mathbf{u}$ and $\mathbf{y}$ can be reduced accordingly.

Example: For the block part example, datum surface $\mathbf{S}_3$ is machined in Stage 1 and later used in Stages 2 and 3. For the two process routes shown in Fig. 4, they use the datum machined by the same machine tool at Stage 1. The machining operations at Stage 2 do not affect $\mathbf{y}_3^{(i)}$ and $\mathbf{y}_3^{(j)}$ because of no datum change. Therefore, two vectors $\mathbf{y}_3^{(i)}$ and $\mathbf{y}_3^{(j)}$ can be reduced into one, e.g., $\mathbf{y}_3^{(i)}$. The corresponding rows in $\boldsymbol{\Gamma}$ can also be eliminated.

By implementing the approaches presented in Sections 3.1, 3.2, and 3.3, dimensions of system model (3) can be reduced. The concept of *minimal* dimension of a model is proposed.

Definition 3.1 A system model has *minimal* dimension if no $\mathbf{u}$ or $\mathbf{y}$ reduction can be further performed on it.

Example: By using $\mathbf{u}$ and $\mathbf{y}$ reductions (Table 3), the system model of block part machining system can be refined. The $\mathbf{u}_{72\times 1}$ is reduced almost by half to $\mathbf{u}_{32\times 1}$, and $\mathbf{y}_{18\times 1}$ is reduced by one-third to $\mathbf{y}_{12\times 1}$. Correspondingly $\boldsymbol{\Gamma}_{18\times 72}$ is significantly reduced to $\boldsymbol{\Gamma}_{12\times 32}$. The refined model has minimal dimension.

Here are two remarks regarding $\mathbf{y}$ reduction and minimal model dimension:

R4 We can treat $\mathbf{y}$ reduction as an extension of $\mathbf{u}$ reduction. It conducts $\mathbf{u}$ reduction first and then delete identical $\mathbf{y}_k^{(j)}$'s in $\mathbf{y}$ and corresponding rows in $\boldsymbol{\Gamma}$.

R5 For a model to be in minimal dimension, the maximum dimension of $\mathbf{u}$ should be $p\Sigma_{k=1}^{N}n_k$, where $p=\mathrm{Dim}(\mathbf{u}_k^{(i)})$, i.e., the dimension of any $\mathbf{u}_k^{(i)}$. For instance, $\mathrm{Dim}(\mathbf{u}_k^{(i)})=4$ in block part example, and $p\Sigma_{k=1}^{N}n_k=4(3+2+3)=32$. This rule can be used to check the accuracy of $\mathbf{u}$ reduction procedure.

4 Impacts of u and y Reduction on System Representation and System Diagnosability

The developed system model is expected to describe the variation streams of all process routes in the system. The refined model should carry the same amount of information as (3). Secondly, the reduction procedures should not worsen the condition of identifying root cases. The former requirement is related to system representation issue, while the latter is about diagnosability, i.e., the ability to identify root causes. For a SP-MMS to be diagnosable, input vector $\mathbf{u}$ in (3) should be estimable with given measurement strategy $\mathbf{y}$. The system diagnosability can be determined by studying the rank of matrix $\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\Gamma}$ [11,16]. The impacts of $\mathbf{u}$ and $\mathbf{y}$ reduction on system representation and diagnosability are addressed in this section.

Claim 4.1 The system model with minimal dimension contains the product variation streams of all process routes in the system.

The techniques of $\mathbf{u}$ and $\mathbf{y}$ reductions aim at eliminating the redundant variables, whose procedures guarantee the claim.

Definition 4.1 The rank deficiency index $\mathrm{I_d}$ is defined as

$$\mathrm{I_d} = \mathrm{Dim}(\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\Gamma}) - \mathrm{Rank}(\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\Gamma}), \tag{12}$$

which represents rank needed to make $\mathbf{u}$ fully estimable or diagnosable.

Since the $\mathrm{I_d}$ index represents the amount of information lacking for the system to be diagnosable, a larger $\mathrm{I_d}$ would suggest that the system is less diagnosable. Based on this criterion, following theorem holds:

Theorem 4.1 The system diagnosability would not decrease by performing $\mathbf{u}$ and $\mathbf{y}$ reduction procedures on system model (3).

Proof: To prove the theorem, we need to prove that the change of $\mathrm{I_d}$ should be less or equal than 0 after $\mathbf{u}$ and $\mathbf{y}$ reduction, i.e., $\Delta\mathrm{I_d} \leqslant 0$. During process design and planning phase, the quality characteristics to be measured are determined as if there were only one process route, e.g., $\mathbf{y}_{\cdot}^{(1)}$. Let the size of $\boldsymbol{\Gamma}_{\cdot}$ be $n \times m$. Assume the rank of $\boldsymbol{\Gamma}_{\cdot}$ is r, i.e., $\mathrm{rank}(\boldsymbol{\Gamma}_{\cdot}) = r$, $r \leqslant \min(n,m)$. The matrix $\boldsymbol{\Gamma}_{\cdot}$ can be transformed into $(\mathbf{G}^{\mathrm{T}},\mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$ by multiplying a matrix $\mathbf{E}$ on the left, where $\mathbf{E}$ is the product of series of elementary matrices. $\mathbf{G}$ is an $r \times m$ matrix. By the definition of $\boldsymbol{\Gamma}$, the following transformation can be made on model (3):

$$\mathrm{diag}(\mathbf{E},\mathbf{E},\ldots,\mathbf{E})\mathbf{y} = \mathrm{diag}(\mathbf{E},\mathbf{E},\ldots,\mathbf{E})\boldsymbol{\Gamma}\mathbf{u} = \begin{pmatrix} \mathbf{G}_{r\times m} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{r\times m} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_{r\times m} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}\mathbf{u} \tag{13}$$

Since term $\boldsymbol{\Gamma}_0\mathbf{x}_0 + \boldsymbol{\varepsilon}$ does not affect system diagnosability, it is dropped in (13) for simplicity.

Denote the transformed $\mathbf{y}$ and $\boldsymbol{\Gamma}$ as $\mathbf{y_E}$ and $\boldsymbol{\Gamma}_{\mathbf{E}}$. Both in $\mathbf{y_E}$ and $\boldsymbol{\Gamma}_{\mathbf{E}}$, rows containing only zeros can be deleted and the deletion does not change the rank of $\boldsymbol{\Gamma}_{\mathbf{E}}$. Instead of analyzing the rank change of $\boldsymbol{\Gamma}$ after $\mathbf{u}$ and $\mathbf{y}$ reduction, we can study $\boldsymbol{\Gamma}_{\mathbf{E}}$ (after deletion).

Not losing generality, suppose $\mathbf{u}$ reduction is performed on the first two blocks in $\boldsymbol{\Gamma}_{\mathbf{E}}$, i.e., $\mathrm{diag}(\mathbf{G}_{\mathrm{r\times m}},\mathbf{G}_{\mathrm{r\times m}})$. Choose r independent columns of $\mathbf{G}$ which spans the range of $\mathbf{G}$, i.e., range $(\mathbf{G}) = \mathrm{span}\{\mathbf{g}_1,\mathbf{g}_2,\ldots,\mathbf{g}_r\}$. Then $\binom{\mathbf{g}_1}{\mathbf{0}}$, $\binom{\mathbf{g}_2}{\mathbf{0}},\ldots,\binom{\mathbf{g}_r}{\mathbf{0}}$, $\binom{\mathbf{0}}{\mathbf{g}_1}$, $\binom{\mathbf{0}}{\mathbf{g}_2},\ldots,\binom{\mathbf{0}}{\mathbf{g}_r}$ are basis for $\mathrm{diag}(\mathbf{G}_{\mathrm{r\times m}},\mathbf{G}_{\mathrm{r\times m}})$. After performing $\mathbf{u}$ reduction on $\mathrm{diag}(\mathbf{G}_{\mathrm{r\times m}},\mathbf{G}_{\mathrm{r\times m}})$, the new matrix is denoted as $\boldsymbol{\Gamma}^{\mathbf{u}}$.

There are four possibilities when performing $\mathbf{u}$ reduction:

(i) $\binom{\mathbf{0}}{\mathbf{g}_j}$ is deleted and $\binom{\mathbf{g}_i}{\mathbf{0}}$ is replaced with $\binom{\mathbf{g}_i}{\mathbf{g}_j}$ $(1 \leqslant i,\ j \leqslant r)$.

The rank of $\boldsymbol{\Gamma}^{\mathbf{u}}$ is either $2r-1$ or $2r$. The dimension of $\boldsymbol{\Gamma}^{\mathbf{u}}$ is $(2r,2m-1)$. Then $\Delta\mathrm{I_d} = [(2m-1)-(2r-1)]-(2m-2r) = 0$, or $\Delta\mathrm{I_d} = [(2m-1)-2r]-(2m-2r) = -1$, i.e., the rank deficiency remains the same or decreases. Here is a physical explanation on when the rank of $\boldsymbol{\Gamma}^{\mathbf{u}}$ is either $2r-1$ or $2r$, i.e., $\Delta\mathrm{I_d} = 0$ or -1.

R6 Suppose routes i and j $(i<j)$ merge at stage k, i.e., $\mathbf{u}_k^{(i)} = \mathbf{u}_k^{(j)}$. If both $\mathbf{u}_k^{(i)}$ and $\mathbf{u}_k^{(j)}$ could be estimated under a given measurement scheme T_0, deletion of $\mathbf{u}_k^{(j)}$ results in rank $(\boldsymbol{\Gamma}^{\mathbf{u}}) = 2r-1$ because only one of them will be estimated in the reduced model. The accuracy of estimating $\mathbf{u}_k^{(i)}$ will be increased too (See **R2**). If only one of $\mathbf{u}_k^{(i)}$ and $\mathbf{u}_k^{(j)}$, or none of them could be estimated under T_0, rank $(\boldsymbol{\Gamma}^{\mathbf{u}})$ remains $2r$ after deleting $\mathbf{u}_k^{(j)}$ and there is no improvement in estimation accuracy.

(ii) $\binom{\mathbf{0}}{\mathbf{g}_j}$ is deleted and a dependent row $\binom{\mathbf{g}'}{\mathbf{0}}$ is replaced with $\binom{\mathbf{g}'}{\mathbf{g}_j}$ $(1 \leqslant j \leqslant r)$.

Since $\mathbf{g}'$ can be represented by a linear combination of $\{\mathbf{g}_1,\mathbf{g}_2,\ldots,\mathbf{g}_{\mathrm{r}}\}$, the new column $\binom{\mathbf{g}'}{\mathbf{g}_j}$ is equivalent to $\binom{\mathbf{0}}{\mathbf{g}_j}$ in terms of serving as a vector in the basis. Therefore, the basis remains the same for $\boldsymbol{\Gamma}^{\mathbf{u}}$. Then $\Delta\mathrm{I_d} = [(2m-1)-2r]-(2m-2r) = -1$, i.e., the rank deficiency is reduced.

(iii) A dependent row $\binom{\mathbf{0}}{\mathbf{g}'}$ is deleted and $\binom{\mathbf{g}_i}{\mathbf{0}}$ is replaced with $\binom{\mathbf{g}_i}{\mathbf{g}'}$ $(1 \leqslant i \leqslant r)$.

Use the same argument in (ii) and $\Delta\mathrm{I_d} = -1$.

(iv) A dependent row is deleted and added to another dependent row.

Since the basis does not change, $\Delta\mathrm{I_d} = -1$.

The results hold for a series of $\mathbf{u}$ reduction and for $\mathbf{y}$ reduction ($\mathbf{y}$ reduction does not reduce the rank of $\boldsymbol{\Gamma}^{\mathbf{u}}$). ■

Example: For the example shown in (8) and (11), $\mathrm{I_d} = 24-6 = 18$ before model refinement. After $\mathbf{u}$ reduction in (8), $\mathrm{I_d} = 16-4 = 12$ and $\Delta\mathrm{I_d} = -6$. After $\mathbf{y}$ reduction, $\mathrm{I_d} = 16-4 = 12$, same as $\mathbf{u}$ reduction. It is interesting to compare $\mathrm{I_d}$ of a system to a single process route. From (6), $\mathrm{I_d}$ of a process route is $\mathrm{I_d} = 12-3 = 9$, which is one half of 18 ($\mathrm{I_d}$ before model refinement). The result can be explained by Theorem 4.2.

Theorem 4.2 For a SP-MMS with R process routes, $\mathrm{I_d}$ of a system model with minimal dimension is bounded,

$$D \leqslant \mathrm{I_d} \leqslant RD, \tag{11}$$

where D denotes $\mathrm{I_d}$ of a single process route.

Proof: Before model refinement, the rank deficiency of (3) is RD, because (3) is a stackup of every process routes. Theorem 4.1 suggests that $\mathrm{I_d} \leqslant RD$. For the system model with minimal dimension, there exists at least one process route (by Claim 4.1). Therefore, the lowest rank deficiency is D for this system model with minimal dimension. ■

Theorem 4.2 suggests that the condition of estimating the root cause in a SP-MMS is worse than any of a single process route in that system. This conclusion is expected because more candidate variation sources need to be estimated.

5 The Implication of u and y Reductions on Measurement Strategy

Since multiple process routes exist in SP-MMSs, measurement strategy is necessary to determine from which routes parts should be measured.

From Theorem 4.2, $\mathrm{I_d}$ of a system model with minimal dimension will be zero if $D = 0$, i.e., the system is fully diagnosable if the process route is diagnosable. Meanwhile, the system model (3), which is not minimal in dimension, is also diagnosable because of $RD = 0$. As mentioned in Section 3, $\mathbf{u}$ and $\mathbf{y}$ reduction is necessary to eliminate redundant variables in the model. This section discusses the advantages to perform model refinement for measurement strategy.

Suppose there are four measurement strategies for the SP-MMS illustrated in Fig. 1:

T1. Measure $\mathbf{y}$ in model (3), i.e., measure $\mathbf{y}_{\cdot}^{(i)}$ for each process route, $i = 1,2,\ldots,R$.

T2. Measure the output $\mathbf{y}_k^{(i)}$ from each machine tool at each stage, $k = 1,2,\ldots,N$. This strategy is commonly applied in a machining system.

T3. Measure $\mathbf{y}$ in the system model with minimal dimension.

T4. Measure $\mathbf{y}_{\cdot}^{(1)}$, $\mathbf{y}_{\cdot}^{(3)}$, and $\mathbf{y}_{\cdot}^{(6)}$.

Assume $D = 0$ and the four strategies T1∼T4 are compared from two aspects: the amount of measurement efforts and the

Table 4 Comparison of measurement strategies T1∼T4

	Amount of Measurement	Sufficiency
T1	$\mathbf{y}_{\cdot}^{(1)}, \mathbf{y}_{\cdot}^{(2)}, \mathbf{y}_{\cdot}^{(3)}, \mathbf{y}_{\cdot}^{(4)}, \mathbf{y}_{\cdot}^{(5)}, \mathbf{y}_{\cdot}^{(6)}$	Yes for MP&IRC
T2	Stage 1: $\mathbf{y}_1^{(1)}, \mathbf{y}_1^{(2)}(=\mathbf{y}_1^{(3)}=\mathbf{y}_1^{(4)}), \mathbf{y}_1^{(5)}(=\mathbf{y}_1^{(6)})$ Stage 2: $\mathbf{y}_2^{(1)}, \mathbf{y}_2^{(2)}(=\mathbf{y}_2^{(3)}=\mathbf{y}_2^{(4)})/\mathbf{y}_2^{(5)}(=\mathbf{y}_2^{(6)})$ Stage 3: $\mathbf{y}_3^{(1)}/\mathbf{y}_3^{(2)}, \mathbf{y}_3^{(3)}/\mathbf{y}_3^{(5)}, \mathbf{y}_3^{(4)}/\mathbf{y}_3^{(6)}$	No for MP&IRC
T3	$\mathbf{y}_1^{(1)}, \mathbf{y}_1^{(2)}(=\mathbf{y}_1^{(3)}=\mathbf{y}_1^{(4)}), \mathbf{y}_1^{(5)}(=\mathbf{y}_1^{(6)})$ $\mathbf{y}_2^{(1)}, \mathbf{y}_2^{(2)}(=\mathbf{y}_2^{(3)}=\mathbf{y}_2^{(4)}), \mathbf{y}_2^{(5)}(=\mathbf{y}_2^{(6)}),$ $\mathbf{y}_3^{(1)}, \mathbf{y}_3^{(2)}, \mathbf{y}_3^{(3)}, \mathbf{y}_3^{(5)}, \mathbf{y}_3^{(4)}, \mathbf{y}_3^{(6)}$	Yes for MP&IRC
T4	$\mathbf{y}_1^{(1)}, \mathbf{y}_1^{(2)}(=\mathbf{y}_1^{(3)}=\mathbf{y}_1^{(4)}), \mathbf{y}_1^{(5)}(=\mathbf{y}_1^{(6)})$ $\mathbf{y}_2^{(1)}, \mathbf{y}_2^{(2)}(=\mathbf{y}_2^{(3)}=\mathbf{y}_2^{(4)}), \mathbf{y}_2^{(5)}(=\mathbf{y}_2^{(6)}),$ $\mathbf{y}_3^{(1)}, \mathbf{y}_3^{(3)}, \mathbf{y}_3^{(6)}$	Yes for IRC but No for MP

sufficiency of information to monitor all process streams (MP) and identify the root causes (IRC). As shown in Table 4, T1 requires the largest amount of measurement effort. Although it is sufficient, it is not economical. T2 ignores the variation streams in different process routes and randomly select parts from each machine tool. This strategy is not effective for statistical analysis because the collected data might come from different distributions. For instance, the part measured at machine 2 of stage 2 could be $\mathbf{y}_2^{(2)}$ or $\mathbf{y}_2^{(5)}$, which represent different variation streams. The data could be insufficient for both process monitoring and root cause diagnosis due to the randomness. Therefore, strategy T2 is least preferable. The amount of measurement for T3 is less than T1 because of $\mathbf{u}$ and $\mathbf{y}$ reduction procedures. It is optimal in the sense of all the variation streams can be captured based on the given measurement. Comparing T3 with T4, T4 requires less measurement and it is still sufficient to estimate root causes $\mathbf{u}$. However, T4 is not sufficient to describe all the variation streams in the system.

Definition 5.1 Type I Set, denoted as S_I, is defined as the minimal set of measurement $\mathbf{y}$ that is sufficient to describe the variation streams in a SP-MMS.

The set of measurement in strategy T3 by definition belongs to S_I. By Claim 4.1, S_I equals to the $\mathbf{y}$ in the system model with minimal dimension.

Definition 5.2 Type II Set, denoted as S_{II}, is defined as the minimal set of measurement $\mathbf{y}$ that is sufficient to estimate $\mathbf{u}$ in the system model with minimal dimension.

The set of measurement in strategy T4 by definition belongs to S_{II}. The relationship between above two measurement sets is established by Theorem 5.1.

Theorem 5.1 If $D=0$, then $S_{II} \subseteq S_I$.

Proof: Since every single machine tool is utilized in manufacturing in a SP-MMS, S_{II} by definition contains the minimal number of process routes which go through all the machine tools. By Claim 4.1, the process routes in S_{II} compose of a subset of the routes in S_I, i.e., $S_{II} \subseteq S_I$. ■

The result suggests that less measurement is required to diagnose root causes than to monitor all variation streams in a SP-MMS.

If $D \neq 0$, the conclusion does not hold because S_{II} requires more information for the system to be diagnosable. Then S_I and S_{II} are not comparable.

6 Conclusion

This paper developed a generic system-level methodology to model the multiple variation streams in a SP-MMS. The state space modeling approach for single process route was extended to the SP-MMS. To identify the redundant variables caused by process coupling, $\mathbf{u}$ and $\mathbf{y}$ reduction techniques were proposed to eliminate the variables in both input vector and output vector. As such, system model with minimal dimension could be obtained to describe all variation streams. As proved in the paper, $\mathbf{u}$ and $\mathbf{y}$ reduction techniques do not affect the model to capture the variation streams. Based on rank deficiency index, $\mathbf{u}$ and $\mathbf{y}$ reduction was proved not to increase $\mathbf{I}_d$, i.e., the system diagnosability would not decrease. It was also proved that the rank deficiency of system model with minimal dimension is bounded.

These results are applied to evaluate different system measurement strategies. It was identified that less measurement is required to diagnose root causes than to monitor all variation streams in a SP-MMS, if single process route is fully diagnosable. Two optimal sets are defined and their relationship was established when the process route is diagnosable.

The methodology development is demonstrated by using simple examples. Efforts have been made to implement the modeling technique in an engine machining plant with success. Future research efforts can be devoted to study system root cause diagnosis and optimal measurement strategy for a SP-MMS. Parameter estimation approaches, such as least square estimation, can be readily applied therein, because the relationship between measurement $\mathbf{y}$ and root causes $\mathbf{u}$ has been established through a linear model.

Acknowledgment

This work is partially supported by NSF Engineering Research Center for Reconfigurable Machining Systems (NSF Grant EEC95-92125) at the University of Michigan and by the University of South Florida, Internal Award Program. The authors are also thankful for the discussion with Prof. Zhifang Zhang at the University of Michigan.

References

[1] Koren, Y., Hu, S. J., and Weber, T., 1998, ‘‘Impact of Manufacturing System Configuration on Performance,’’ CIRP Ann., **47**, pp. 369–372.
[2] Boyd, D. F., 1950, ‘‘Applying the Group Control Chart for $\bar{x}$ and *R*,’’ Industrial Quality Control, **3**, pp. 22–25.
[3] Mortell, R. R., and Runger, G. C., 1995, ‘‘Statistical Process Control of Multiple Stream Processes,’’ Journal of Quality Technology, **27**, pp. 1–12.
[4] Jin, J., and Shi, J., 1999, ‘‘State Space Modeling of Sheet Metal Assembly for Dimensional Control,’’ ASME J. Manuf. Sci. Eng., **121**, pp. 756–762.
[5] Mantripragada, R., and Whitney, D. E., 1999, ‘‘Modeling and Controlling Variation Propagation in Mechanical Assemblies Using State Transition Models,’’ IEEE Trans. Rob. Autom., **15**, pp. 124–140.
[6] Lawless, J. F., Mackay, R. J., and Robinson, J. A., 1999, ‘‘Analysis of Variation Transmission in Manufacturing Processes—Part I,’’ Journal of Quality Technology, **31**, pp. 131–142.
[7] Agrawal, R., Lawless, J. F., and Mackay, R. J., 1999, ‘‘Analysis of Variation Transmission in Manufacturing Processes—Part II,’’ Journal of Quality Technology, **31**, pp. 143–154.
[8] Huang, Q., Shi, J., and Yuan, J., 2003, ‘‘Part Dimensional Error and its Propagation Modeling in Multi-Operational Machining Processes,’’ ASME J. Manuf. Sci. Eng., **125**, pp. 255–262. Also appeared in *Proc. 2000 ASME Int. Mech. Eng. Congress & Exposition, MED-Vol.*, **11**, pp. 81–88.
[9] Djurdjanovic, D., and Ni, J., 2001, ‘‘Linear State Space Modeling of Dimensional Machining Errors,’’ NAMRI/SME, **XXIX**, pp. 541–548.
[10] Zhou, S., Huang, Q., and Shi, J., 2003, ‘‘State Space Modeling for Dimensional Monitoring of Multistage Machining Process Using Differential Motion Vector,’’ IEEE Trans. Rob. Autom., **19**, pp. 296–309.
[11] Ding, Y., Ceglarek, D., and Shi, J., 2002, ‘‘Fault Diagnosis of Multistage Manufacturing Processes by Using State Space Approach,’’ ASME J. Manuf. Sci. Eng., **124**, pp. 313–322.
[12] Huang, Q., and Shi, J., 2004, ‘‘Variation Transmission Analysis and Diagnosis of Multi-Operational Machining Processes,’’ IIE Transactions, Quality and Reliability Engineering, **36** (in press).
[13] Martinsen, K., 1993, ‘‘Vectorial Tolerancing for All Types of Surfaces,’’ *ASME Advances in Design Automation*, **2**, pp. 187–198.
[14] Kuo, B. C., 1977, *Digital Control Systems*, SRL Publishing Company, Champaign, Illinois, pp. 148–151.
[15] Huang, Q., and Shi, J., 2003, ‘‘Simultaneous Tolerance Synthesis Through Variation Propagation Modeling of Multistage Manufacturing Processes,’’ NAMRI/SME Transactions, **XXXI**, pp. 515–522.
[16] Wang, Y., and Nagarkar, S. R., 1999, ‘‘Locator and Sensor Placement for Automated Coordinate Checking Fixtures,’’ ASME J. Manuf. Sci. Eng., **121**, pp. 709–719.