Appears in the Proceedings of the 8th International Symposium on Communication Theory and Applications (ISCTA), July 2005

Continuous Phase Modulated Space-Time Codes

Alenka G. Zajić and Gordon L. Stüber
School of Electrical and Computer Engineering
Georgia Institute of Technology,Atlanta, GA 30332 USA

***Abstract*— Conditions are derived under which M-ary full-response CPM space-time codes will attain full spatial diversity. General code construction rules are desirable due to the nonlinearity and inherent memory of CPM signals which make manual design or computer search difficult. A linear decomposition of CPM signals is used to identify a rank criterion for M-ary full-response CPM that specifies the set of allowable modulation indices.**

Keywords: space-time coding, continuous phase modulation, rank criterion, spatial diversity, Rayleigh fading.

I. Introduction

Transmit and receive diversity techniques are commonly used in wireless systems. Space-time coding is a transmit diversity technique that exploits a dense scattering environment by using multiple transmit antennas [1]– [7]. It is desirable that the space-time codes attain *full spatial diversity* so that all channels in the multi-input multi-output (MIMO) system contribute an independent spatial diversity gain. Tarokh *et al.* [1] devised a rank criterion for spatial diversity that optimizes the worst case pair-wise error probability (PWEP), and presented some simple design rules that guarantee full diversity for linear modulation schemes. Hammons and Gamal [2] proposed rank criteria that ensure full diversity for space-time codes with binary phase shift keying (BPSK) and quaternary phase shift keying (QPSK). Liu *et al.* [3] generalized these rank criteria for higher-order quadrature amplitude modulation (QAM) constellations.

Continuous phase modulation (CPM) is a nonlinear modulation scheme with high bandwidth efficiency [8]. CPM has a constant envelope and is thus very useful for radio systems that employ nonlinear amplifiers. Combining CPM with space-time coding (ST-CPM) is a natural extension of space-time coding with linear modulation. Space-time code design for CPM is more difficult than for linear modulation due to nonlinearity of the modulation and its more complex performance matrices. It is desirable to define some general code constructions which can guarantee full diversity with CPM. Recently, Zhang and Fitz [4], [5] have studied space-time coding techniques for CPM over quasi-static fading channels. They derived design criteria for ST-CPM with quasi-static fading and, similarly to linear modulation, defined the signal matrix associated with the PWEP. They also identified the rank criterion for certain CPM schemes: full response 2^n-ary CPM with $h = 1/2$, full response 4^n-ary CPM with $h = 1/4$, partial response binary CPM with $h = 1/2$ and partial response 4-ary CPM with $h = 1/4$.

This paper investigates conditions under which M-ary full-response CPM space-time codes will attain full diversity. Section II introduces the space-time coded CPM system, while Section III reviews Mengali and Morelli's linear decomposition for full-response CPM signals [9]. Section IV is the central contribution of the paper and presents our rank criterion for M-ary full-response CPM that specifies the set of allowable modulation indices. Section V includes some verification examples while Section VI concludes the paper.

Prepared through collaborative participation in the Collaborative Technology Alliance for Communications & Networks sponsored by the U.S. Army Research Laboratory under Cooperative Agreement DAAD19-01-2-0011. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation thereon.

II. Space-Time Coded CPM System

This paper considers a space-time coded continuous phase modulated (ST-CPM) system with L_t transmit antennas and L_r receive antennas. As shown in Fig. 1, the K_b information symbols are input to a space-time (ST) encoder. The ST encoder uses the error control code $\mathcal{C}$ to encode information symbols into codeword vectors $\hat{\mathbf{c}} \in \mathcal{C}$ of length $N = N_c L_t$ and map them onto an $L_t \times N_c$ matrix $\mathbf{C}$. Mapping is done in the following manner: codeword

$$\hat{\mathbf{c}} = \left(c_1^{(1)}, \ldots, c_{N_c}^{(1)}, \ldots c_1^{(L_t)}, \ldots, c_{N_c}^{(L_t)}\right) \quad (1)$$

is mapped onto the $L_t \times N_c$ matrix

$$\mathbf{C} = \begin{bmatrix} c_1^{(1)} & \cdots & c_{N_c}^{(1)} \\ \vdots & \ddots & \vdots \\ c_1^{(L_t)} & \cdots & c_{N_c}^{(L_t)} \end{bmatrix}, \quad (2)$$

where $c_k^{(i)}$ is the code symbol assigned to i-th transmit antenna at time epoch k.

The outputs of the space-time encoder are L_t streams of symbols. Each stream of symbols after modulation mapping is input to a CPM modulator. The CPM modulated signals are simultaneously transmitted from L_t transmit antennas.

The received signal at each receive antenna is a noisy superposition of L_t transmitted signals, each corrupted by quasi-static flat Rayleigh fading, and independent zero-mean complex additive white Gaussian noise (AWGN). With these assumptions, the signal received by antenna j can be written as [10]

$$r_j(t) = \sum_{i=1}^{L_t} \alpha_{i,j} s_i(t) + n_j(t), \quad (3)$$

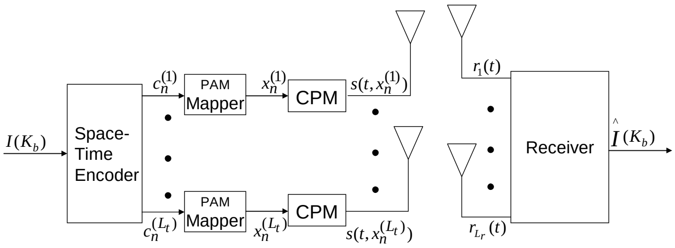

Fig. 1. Space-time coded CPM system.

where $r_j(t)$ is the signal received at antenna j at time t, and $\alpha_{i,j}$ is the complex path gain from transmit antenna i to receive antenna j at time t. The noise samples are independent samples of a zero-mean complex Gaussian random variable with variance $N_0/2$ per dimension. If we define vectors

$$\mathbf{s}(t) = \begin{bmatrix} s_1(t) \\ \vdots \\ s_{L_t}(t) \end{bmatrix}, \ \mathbf{r}(t) = \begin{bmatrix} r_1(t) \\ \vdots \\ r_{L_r}(t) \end{bmatrix}, \ \mathbf{n}(t) = \begin{bmatrix} n_1(t) \\ \vdots \\ n_{L_r}(t) \end{bmatrix}, \quad (4)$$

the received signal can be represented in the vector form

$$\mathbf{r}(t) = \mathbf{A}^\top \mathbf{s}(t) + \mathbf{n}(t), \quad (5)$$

where $\mathbf{A} = [\alpha_{i,j}]_{L_t \times L_r}$ is the matrix of complex fading gains.

The optimum receiver for ST-CPM employs maximum-likelihood sequence detection (MLSD). The typical method for analyzing the performance with MLSD is to evaluate an upper bound on the pairwise error probability for any two space-time codewords $\mathbf{c}$ and $\hat{\mathbf{c}}$, presented in matrix form $\mathbf{C}$, and $\hat{\mathbf{C}}$, as defined in (2). Let $\mathbf{s}(t)$ and $\hat{\mathbf{s}}(t)$ denote the CPM signals corresponding to codewords $\mathbf{C}$ and $\hat{\mathbf{C}}$, respectively. The pairwise error probability is [2]

$$P_e(\mathbf{C}, \hat{\mathbf{C}}|\mathbf{A}) = P(\mathbf{s}(t) \rightarrow \hat{\mathbf{s}}(t)|\mathbf{A}) = Q\left(\frac{\| \mathbf{A}^\top \Delta\mathbf{s}(t) \|}{\sqrt{2N_0}}\right) \quad (6)$$

where $\Delta\mathbf{s}(t) = \mathbf{s}(t) - \hat{\mathbf{s}}(t)$. For a quasi-static flat Rayleigh fading channel, (6) can be manipulated to yield the following upper bound on pairwise error probability [2]

$$\begin{aligned} P_e(\mathbf{C}, \hat{\mathbf{C}}|\mathbf{A}) &\leq \left(\frac{1}{\prod_{i=1}^{r}(1 + \lambda_i E_s/4N_0)}\right)^{L_r} \\ &\leq \left(\frac{\eta E_s}{4N_0}\right)^{-rL_r}, \end{aligned} \quad (7)$$

where r is the rank of the correlation matrix $\mathbf{C}_\mathbf{s}$, $\{\lambda_1, \cdots, \lambda_r\}$ are the nonzero eigenvalues of $\mathbf{C}_\mathbf{s}$, and $\eta = (\lambda_1\lambda_2 \cdots \lambda_r)^{1/r}$ is their geometric mean. The matrix of correlation functions of the differential CPM signals received at the different antennas $\mathbf{C}_\mathbf{s}$ is equal to [4]

$$\begin{bmatrix} \int_0^{N_cT_c} |\Delta s_1(t)|^2 dt & \cdots & \int_0^{N_cT_c} \Delta s_1(t)\Delta s^*_{L_t}(t)dt \\ \vdots & \vdots & \vdots \\ \int_0^{N_cT_c} \Delta s_{L_t}(t)\Delta s^*_1(t)dt & \cdots & \int_0^{N_cT_c} |\Delta s_{L_t}(t)|^2 dt \end{bmatrix}. \quad (8)$$

Equation (7) shows that the design and performance of ST-CPM has a direct analogy to the design and performance of space-time coded linear modulation. Consequently, the rank criterion and the product distance criterion for space-time linear modulation are directly applicable to ST-CPM, the only difference being the "signal" matrix (8).

III. Review of CPM Signals

The CPM complex envelope with normalized amplitude can be represented as

$$s(t) = e^{j\phi(t;\mathbf{x})}, \quad (9)$$

where

$$\phi(t;\mathbf{x}) = 2\pi h \sum_{k=0}^{N_c-1} x_{k+1} q(t - kT_c) \quad (10)$$

is the excess phase, h is the modulation index, $\mathbf{x} = (x_0, \cdots, x_{N_c-1})$ is the information sequence with elements chosen from the M-ary alphabet $\{\pm1, \pm3, \cdots, \pm(M-1)\}$, and T_c is the symbol duration. The function $q(t)$ is the phase shaping pulse defined by $q(t) = \int_0^t g(\tau)d\tau$, and $g(t)$ is the frequency shaping pulse of length LT_c, such that $q(LT_c) = 1/2$. The CPM waveform has full-response when $L = 1$ and partial-response when $L > 1$.

Mengali and Morelli [9] showed that M-ary full-response CPM signals can be represented as

$$s(t;\mathbf{x}) = \sum_{k=0}^{R-1} \sum_{n=0}^{N_c-1} B_{k,n} g_k(t - nT_c), \quad (11)$$

where

$$g_k(t) = \prod_{l=0}^{F-1} c_0^{(l)} \left[t + m_l T_c\right], \quad (12)$$

$$B_{k,n} = \exp\left(j\pi h \sum_{l=0}^{F-1} 2^l A_{0,n-m_l}^{(l)}\right), \quad (13)$$

$$c_0^{(l)}(t) = \begin{cases} \dfrac{\sin\left(2\pi h 2^l q(t)\right)}{\sin(\pi h 2^l)} & , \ 0 \leq t < T_c \\ c_0^{(l)}(2T_c - t) & , \ T_c \leq t \leq 2T_c \\ 0 & , \ \text{otherwise.} \end{cases} \quad (14)$$

and $R = (2^F - 1)$, $2^{F-1} \leq M \leq 2^F$, $0 \leq m_l \leq T_l - 1$, and $\prod_{l=0}^{F-1} m_l = 0$, where T_l is the duration of signal $c_0^{(l)}(t)$.

The coefficient $A_{0,n-m_l}^{(l)}$ is defined as

$$A_{0,n-m_l}^{(l)} = \sum_{m=0}^{n-m_l} \gamma_{m,l}, \quad (15)$$

where $\gamma_{m,l}$ are binary coefficients of the information symbol x_n. Signal $B_{k,n}$ can be further modified to

$$B_{k,n} = \exp\left[j\pi h\left(\sum_{r=0}^{n-m_l} 2u_r\right)\right]\exp\left(-j\pi h d_n\right), \quad (16)$$

where $u_r = \sum_{l=0}^{F-1} 2^{l-1}(\gamma_{m,l}+1)$ and $d_n = \sum_{l=0}^{F-1} 2^{l}(n - m_l + 1)$.

IV. A Rank Criterion for M-ary Full-Response CPM Space-Time Codes

The inputs to CPM modulators in Fig. 1 are elements from the $L_t \times N_c$ matrix

$$\mathbf{X} = \begin{bmatrix} x_1^{(1)} & \cdots & x_{N_c}^{(1)} \\ \vdots & \ddots & \vdots \\ x_1^{(L_t)} & \cdots & x_{N_c}^{(L_t)} \end{bmatrix}, \quad (17)$$

where $x_k^{(i)}$ is the PAM mapped symbol assigned to i-th transmit antenna at time epoch k. The outputs from the CPM modulators are signals $s(t;\mathbf{x}^{(i)})$ as defined in (11). Assume that $h = K/P$, where K and P are relatively prime integers. If we define $v_{k,n}^{(i)}$ as

$$v_{k,n}^{(i)} = \left[\sum_{r=0}^{n-m_l} u_r^{(i)}\right]_{\mathrm{mod\,P}}, \quad (18)$$

where $u_r^{(i)} = (x_r^{(i)} - M + 1)/2$ and $x_r^{(i)}$ is element from the matrix $\mathbf{X}$, then $v_{k,n}^{(i)} \in \{0, 1, \cdots, P-1\}$.

Define the matrix of accumulative values as $\mathbf{V}_k \triangleq [v_{k,n}^{(i)}]$ and denote $e^{\mathbf{V}_k} = [e^{j2\pi h v_k^{(i)}}]$. Also, define the diagonal matrix $\mathbf{D} \triangleq [e^{j\pi h d_n}]$ with elements $d_n = \sum_{\ell=0}^{F-1} 2^{\ell}(n - m_\ell + 1)$. Finally, define the vector $\mathbf{g}_k(t) = [g_k(t), g_k(t-T_c), \cdots, g_k(t-(N_c-1)T_c)]^T$. From (11)-(16) and (18), the space-time coded M-ary full-response CPM signal $\mathbf{s}(t;\mathbf{V})$ can be represented as

$$\mathbf{s}(t;\mathbf{V}) = \sum_{k=0}^{R-1} e^{\mathbf{V}_k}\ \mathbf{D}\ \mathbf{g}_k(t) \quad (19)$$

Therefore, the CPM difference signal matrix for two space-time codewords, $\mathbf{C}$ and $\hat{\mathbf{C}}$, is

$$\Delta\mathbf{s}(t) = \sum_{k=0}^{R-1}(e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})\ \mathbf{D}\ \mathbf{g}_k(t). \quad (20)$$

Space-time coded CPM signals will achieve full diversity if the matrix $\mathbf{C}_s$, defined in (8), has full rank. It has been proven in [4] that the matrix $\mathbf{C}_s$ has full rank if and only if all functions in the differential matrix $\Delta\mathbf{s}(t)$, defined in (20), are linearly independent. This will be satisfied if the functions in $\mathbf{g}_k(t)$ are linearly independent and the matrix $(e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$ has full rank L_t for any two space-time codewords. Note that matrix $\mathbf{D}$ has full rank and does not effect the result. Before we state our rank criterion, we introduce several definitions and theorems.

Definition 1: *The set of functions ψ_{i} is defined as ψ_{i} : $\{\psi_{\mathrm{i}}(\mathrm{a}) = \mathrm{Res}[a/2^{i+1}]/2^i\}$, where a is an integer from the set $\{0, 1, \cdots, M-1\}$, and $\mathrm{Res[x/y]}$ denotes residue of x modulo y.*
Note that function $\psi_i(a)$ takes values from the set $\{0, 1\}$. Every integer a from the set $\{0, 1, \cdots, M-1\}$, where $M \leq 2^k$ and k is an integer, can be represented in radix-2 form $a = \sum_{i=0}^{k-1} 2^i \psi_i(a)$.

Lemma 1: *If integers a and b are chosen from the set $\{0, 1, \cdots, M-1\}$, where M is an even integer, k is an integer, and $M \leq 2^k$, then the following equality holds:*

$$(a+b)_{\mathrm{mod\ M}} = (\psi_0(a) + \psi_0(b))_{\mathrm{mod\ 2}} + \sum_{i=1}^{k-1} 2^i(\psi_i(a) + \psi_i(b) + c_{i-1})_{\mathrm{mod\ 2}}, \quad (21)$$

where c_{i-1} denotes carry bit from binary addition.
The proof of this lemma is shown in Appendix A.

Definition 2: *The set of matrix functions Φ_{i} is defined as Φ_{i} : $\left\{\Phi_{\mathrm{i}}(\mathbf{A}) = \mathrm{Res}[\mathbf{A}/2^{i+1}]/2^i\right\}$, where $\mathbf{A}$ is $r \times n$ matrix, $r \leq n$, with elements $a_{i,j} \in \mathbf{A}$ from the set $\{0, 1, \cdots, M-1\}$, where M is an even integer, k is an integer, and $M \leq 2^k$. $\mathrm{Res[x/y]}$ denotes the residue of x modulo y.*
Note that $\Phi_i(A)$ is an $r \times n$ matrix with elements from the set $\{0, 1\}$.

Lemma 2: *Let $\mathbf{A}$ and $\mathbf{B}$ be $r \times n$ matrices, such that elements $a_{i,j} \in \mathbf{A}$ and $b_{i,j} \in \mathbf{B}$ are integers from the set $\{0, 1, \cdots, M-1\}$, where M is an even integer, k is an integer, and $M \leq 2^k$. Define functions $f(\mathbf{A})$ and $g(\mathbf{A})$ as $f(\mathbf{A})$: $\left\{f(a_{i,j}) = \left(\sum_{m=0}^{j-w} a_{i,m}\right)_{\mathrm{mod\ M}}\right\}$ and $g(\mathbf{A})$: $\left\{g(a_{i,j}) = \left(\sum_{m=0}^{j-w} \psi_0(a_{i,m})\right)_{\mathrm{mod\ 2}}\right\}$, where w is an integer and $w \leq j$. If $\mathbf{B} = f(\mathbf{A})$, then $\Phi_0(\mathbf{B}) = g(\Phi_0(\mathbf{A}))$.*
The proof of this lemma is shown in Appendix B.

Lemma 3: *Let $\mathbf{A}$ be $r \times n$ matrix, $r \leq n$, with elements $a_{i,j} \in \mathbf{A}$ from the set $\{0, 1, \cdots, M-1\}$, where M is an even integer, k is an integer, and $M \leq 2^k$. The set of functions $\Phi_i(\mathbf{A})$ is as defined in Definition 2 and the set of functions $\psi_i(a)$ is as defined in Definition 1. If the matrix $\Phi_0(\mathbf{A})$ has full rank, then the matrix $\mathbf{A}$ also has full rank.*
The proof of this lemma is shown in Appendix C.

Lemma 4: *If β is a complex primitive 2^F-th root of unity, and $\mathbb{Z}[\beta]$ denotes the ring of integers in the cyclotomic extension of the rational numbers $\mathbb{Q}$, then*

$$\frac{\beta^{\sum_{k=0}^{F-1} a_k 2^k} - \beta^{\sum_{k=0}^{F-1} b_k 2^k}}{1-\beta} \equiv a_0 \oplus b_0\,[\mathrm{mod}\ (1-\beta)], \quad (22)$$

where $\oplus$ denotes addition modulo 2, and $a_k, b_k \in GF(2)$.

Proof of this lemma is presented in [7].

Lemma 5: *If the complex matrix $\Delta\mathbf{V} = (e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$ has full rank, then the differential matrix $\Delta\mathbf{s}(t)$ has full rank, i.e., the ST-CPM system achieves full spatial diversity L_t.*

The proof of this lemma is shown in Appendix D.

Theorem 6 (A Rank Criterion for M-ary CPM) : *Denote $\mathcal{C}$ as a linear $L_t \times N_c$ space-time code with $L_t \le N_c$. Suppose that all M-ary codewords $\mathbf{C} \in \mathcal{C}$ (where $2^F \le M < 2^{F+1}$) are generated as $\mathbf{C} = \sum_{l=0}^{F-1} 2^l \mathbf{C}_l$, where $\mathbf{C}_l$ are binary codewords from the linear $L_t \times N_c$ space-time codes $\mathcal{C}_l$ and all codewords $\mathbf{C}_0 \in \mathcal{C}_0$ achieve full spatial diversity. Define two codewords $\mathbf{C}, \hat{\mathbf{C}} \in \mathcal{C}$ to be different if $\mathbf{C}_0 \neq \hat{\mathbf{C}}_0$. Then for any full-response M-ary CPM with $h = K/2^n$ (where $2^n \le 2^F$, and 2^n and K are relatively prime integers), the space-time code $\mathcal{C}$ achieves full spatial diversity L_t.*

Proof: In Lemma 5 we proved that the ST-CPM system achieves full spatial diversity if the complex matrix $\Delta\mathbf{V} = (e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$ has full rank. We now show that matrix $\Delta\mathbf{V}$ has full rank if the space-time code $\mathcal{C}$ is as defined in Theorem 6.

First note that codewords $\mathbf{C}_i \in \mathcal{C}_i$ are equal to $\Phi_i(\mathbf{C})$, where functions $\Phi_i(\mathbf{C})$ are defined in Definition 2. Since codeword $\mathbf{C}_0 = \Phi_0(\mathbf{C})$ has full rank, using Lemma 3 we can conclude that $\mathbf{C}$ also has full rank. Earlier we have defined matrix

$$\mathbf{V}_k = [v_{k,n}^{(i)}] = [\sum_{r=0}^{n-m_l} u_r^{(i)}]_{\bmod 2^n}. \tag{23}$$

Note that $\mathbf{V}_k = f(\mathbf{C})$, where function $f(\mathbf{C})$ is defined as $f(\mathbf{C}) : \left\{ f(c_n^{(i)}) = \left[\sum_{r=0}^{n-m_l} u_r^{(i)}\right]_{\bmod 2^n} \right\}$. From Lemma 2, we can conclude that $\Phi_0(\mathbf{V}_k) = g\left(\Phi_0(\mathbf{C})\right)$, where $g\left(\Phi_0(\mathbf{C})\right)$: $\left\{ g(c_0^{(i)}) = \left(\sum_{r=0}^{n-m_l} u_r^{(i)}\right)_{\bmod 2} \right\}$. Since $g\left(\Phi_0(\mathbf{C})\right)$ is a linear operation, $\Phi_0(\mathbf{V}_k)$ also belongs to the code $\mathcal{C}_0$. It follows that $\Phi_0(\mathbf{V}_k)$ has full rank. From Lemma 3 we can conclude that matrix $\mathbf{V}_k$ also has full rank. Note that because $g\left(\Phi_0(\mathbf{C})\right)$ is a linear operation, if $\Phi_0(\mathbf{C}) \neq \Phi_0(\hat{\mathbf{C}})$ then $\Phi_0(\mathbf{V}_k) \neq \Phi_0(\hat{\mathbf{V}}_k)$. This means that the two accumulative matrices $\mathbf{V}_k$ and $\hat{\mathbf{V}}_k$ are different if $\Phi_0(\mathbf{V}_k)$ and $\Phi_0(\hat{\mathbf{V}}_k)$ are different.

The next step is to show that matrix $\Delta\mathbf{V} = (e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$ has full rank for any two different accumulative matrices $\mathbf{V}_k$ and $\hat{\mathbf{V}}_k$. Define $\beta = \exp(j2\pi h) = \exp(j2\pi K/2^n)$ and note that β is a complex, primitive 2^n-th root of unity. It is sufficient to prove that it is not possible to find a nonzero vector $\mathbf{b}$ such that $\mathbf{b}^\top \Delta\mathbf{V} = \mathbf{0}^\top$, where components of vector $\mathbf{b}$ are from $\mathbb{Z}[\beta]$ and not all components of $\mathbf{b}$ are divisible by $(1-\beta)$. We will prove this by contradiction. Assume $\Delta\mathbf{V}$ does not have full rank. From Lemma 4, we have $\Delta\mathbf{V}/(1-\beta) \equiv \Phi_0(\mathbf{V}_k) \oplus \Phi_0(\hat{\mathbf{V}}_k)\,[\mathrm{mod}\ (1-\beta)]$. Thus

$$\mathbf{b}^\top \Delta\mathbf{V} = \mathbf{0}^\top \Leftrightarrow \mathbf{b}^\top \frac{\Delta\mathbf{V}}{1-\beta} = \mathbf{0}^\top \Leftrightarrow$$
$$\mathbf{b_1}^\top \left(\Phi_0(\mathbf{V}_k) \oplus \Phi_0(\hat{\mathbf{V}}_k)\right) (\mathrm{mod}\ (1-\beta)), \tag{24}$$

where $\mathbf{b_1} = \mathbf{b}/(1-\beta)$. It follows that $\mathbf{b_1} = \mathbf{b}\,(\mathrm{mod}\ (1-\beta)) \neq \mathbf{0}$. By linearity, matrix $\Phi_0(\mathbf{V}_k) \oplus \Phi_0(\hat{\mathbf{V}}_k)$ also belongs to code $\mathcal{C}_0$. This implies that matrix $\Delta\mathbf{V}$ has full rank, contradicting our assumption. ⊡

We proved that if the space-time code $\mathcal{C}$ is as defined in Theorem 6, the matrix $\Delta\mathbf{V} = (e^{\mathbf{V}_k} - e_k^{\hat{\mathbf{V}}})$ has full rank for any two different accumulative matrices $\mathbf{V}_k$ and $\hat{\mathbf{V}}_k$. In Lemma 5 we proved that $\Delta\mathbf{s}(t)$ has full rank if matrix $\Delta\mathbf{V}$ has full rank. Consequently, the overall ST-CPM system achieves full diversity.

V. Examples

In this section we construct two codewords from a (3×4) 8-ary full-response CPM space-time code $\mathcal{C}$, as defined in Theorem 6 and verify that ST-CPM system with these codewords achieves full spatial diversity. Every codeword $\mathbf{C} \in \mathcal{C}$ is generated as $\mathbf{C} = \sum_{l=0}^{2} 2^l \mathbf{C}_l$, where $\mathbf{C}_l$ are binary codewords from linear (3×4) space-time codes $\mathcal{C}_l$ and all codewords $\mathbf{C}_0 \in \mathcal{C}_0$ have full rank over GF(2). Some of the methods to construct binary space-time codes with full rank are described in [2] and [6]. Binary codewords $\mathbf{C}_1$ and $\mathbf{C}_2$ can be generated the same way as the codeword $\mathbf{C}_0$, or can be chosen to optimize other code performance criterion, e.g., the coding gain. Two codewords $\mathbf{C}, \hat{\mathbf{C}} \in \mathcal{C}$ are defined to be different if $\mathbf{C}_0 \neq \hat{\mathbf{C}}_0$. Following the method described in [6], we use α as a zero of the primitive polynomial $f(x) = x^4 + x + 1$ over GF(2) and we use the generator matrix $G = [1 \quad \alpha \quad \alpha^2]$ to construct codewords:

$$\mathbf{C} = \begin{bmatrix} 2 & 6 & 5 & 2 \\ 6 & 5 & 2 & 0 \\ 7 & 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \tag{25}$$

and

$$\hat{\mathbf{C}} = \begin{bmatrix} 7 & 6 & 1 & 7 \\ 6 & 1 & 5 & 2 \\ 7 & 5 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{26}$$

Note that binary codewords $\mathbf{C}_1$ and $\hat{\mathbf{C}}_1$ do not have full rank. According to Theorem 6, these two codewords will achieve full spatial diversity, if modulation index h takes values from the set $\{1/2, 1/4, 1/8\}$. We now verify this statement for each h in the set.

First, let $h = 1/8$. To verify that the codewords (25) and (26) achieve full diversity in ST-CPM system, we need to check if all functions in the differential matrix $\Delta\mathbf{s}(t)$, defined in (20), are linearly independent. This will be satisfied if components of the vector $\mathbf{g}_k(t)$ are linearly independent and the matrix

$(e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$ has full rank. From (11)-(16) we calculate functions $g_k(t)$ and modulated symbols $B_{k,n}$, with the results in Table I. Examining functions $g_k(t)$, we can conclude that they are linearly independent. Table II shows the matrices $\mathbf{V}_k$, $\hat{\mathbf{V}}_k$, and $(e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$, where $\mathbf{V}_k$ is defined in (23). Observe that all matrices in Table II have full rank, which implies that the pair of codewords in (25) and (26) achieves full spatial diversity.

TABLE I

CPM SYMBOLS AND LAURENT FUNCTIONS FOR $M = 8, L = 1$

k	$g_k(t)$	$B_{k,n}$
0	$c_0^{(0)}(t)c_0^{(1)}(t)c_0^{(2)}(t)$	$\exp[j\pi h(A_{0,n}^{(0)} + 2A_{0,n}^{(1)} + 4A_{0,n}^{(2)})]$
1	$c_0^{(0)}(t+T_c)c_0^{(1)}(t)c_0^{(2)}(t)$	$\exp[j\pi h(A_{0,n-1}^{(0)} + 2A_{0,n}^{(1)} + 4A_{0,n}^{(2)})]$
2	$c_0^{(0)}(t)c_0^{(1)}(t+T_c)c_0^{(2)}(t)$	$\exp[j\pi h(A_{0,n}^{(0)} + 2A_{0,n-1}^{(1)} + 4A_{0,n}^{(2)})]$
3	$c_0^{(0)}(t+T_c)c_0^{(1)}(t+T_c)c_0^{(2)}(t)$	$\exp[j\pi h(A_{0,n-1}^{(0)} + 2A_{0,n-1}^{(1)} + 4A_{0,n}^{(2)})]$
4	$c_0^{(0)}(t)c_0^{(1)}(t)c_0^{(2)}(t+T_c)$	$\exp[j\pi h(A_{0,n}^{(0)} + 2A_{0,n}^{(1)} + 4A_{0,n-1}^{(2)})]$
5	$c_0^{(0)}(t+T_c)c_0^{(1)}(t)c_0^{(2)}(t+T_c)$	$\exp[j\pi h(A_{0,n-1}^{(0)} + 2A_{0,n}^{(1)} + 4A_{0,n-1}^{(2)})]$
6	$c_0^{(0)}(t)c_0^{(1)}(t+T_c)c_0^{(2)}(t+T_c)$	$\exp[j\pi h(A_{0,n}^{(0)} + 2A_{0,n-1}^{(1)} + 4A_{0,n-1}^{(2)})]$

TABLE II

RANK VERIFICATION FOR FULL-RESPONSE CPM $M = 8, h = 1/8$

k	$V_k - V_k^{'}$	$\Delta V_k = e^{V_k} - e^{V_k^{'}}$
0	$\begin{bmatrix} 2 & 0 & 5 & 7 \\ 6 & 3 & 5 & 5 \\ 7 & 1 & 5 & 3 \end{bmatrix} - \begin{bmatrix} 7 & 5 & 6 & 5 \\ 6 & 7 & 4 & 6 \\ 7 & 4 & 0 & 6 \end{bmatrix}$	$\begin{bmatrix} -0.707+1.707j & 1.707+0.707j & -0.707+0.2929j & 1.4142 \\ 0 & -1.4142+1.4142j & 0.2929-0.707j & -0.707+0.2929j \\ 0 & 1.707+0.707j & -1.707-0.707j & -0.707+1.707j \end{bmatrix}$
1	$\begin{bmatrix} 2 & 0 & 5 & 7 \\ 6 & 3 & 5 & 5 \\ 7 & 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 4 & 5 & 4 \\ 6 & 7 & 4 & 6 \\ 7 & 3 & 7 & 5 \end{bmatrix}$	$\begin{bmatrix} -0.707+1.707j & 2 & 0 & 1.707-0.707j \\ 0 & -1.4142+1.4142j & 0.2929-0.707j & -0.707+0.2929j \\ 0 & 1.707-0.707j & -1.707+0.707j & 0.707+1.707j \end{bmatrix}$
2	$\begin{bmatrix} 2 & 6 & 3 & 5 \\ 6 & 1 & 3 & 3 \\ 7 & 7 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 3 & 4 & 3 \\ 6 & 5 & 2 & 4 \\ 7 & 2 & 6 & 4 \end{bmatrix}$	$\begin{bmatrix} -0.707+1.707j & 0.707-1.707j & 0.2929+0.707j & -1.4142j \\ 0 & 1.4142+1.4142j & -0.707-0.2929j & 0.2929+0.707j \\ 0 & 0.707-1.707j & -0.707+1.707j & 1.707+0.707j \end{bmatrix}$
3	$\begin{bmatrix} 2 & 6 & 3 & 5 \\ 6 & 1 & 3 & 3 \\ 7 & 6 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 7 & 2 & 3 & 2 \\ 6 & 5 & 2 & 4 \\ 7 & 1 & 5 & 3 \end{bmatrix}$	$\begin{bmatrix} -0.707+1.707j & -2j & 0 & -0.707-1.707j \\ 0 & 1.4142+1.4142j & -0.707-0.2929j & 0.2929+0.707j \\ 0 & -0.707-1.707j & 0.707+1.707j & 1.707-0.707j \end{bmatrix}$
4	$\begin{bmatrix} 2 & 0 & 5 & 7 \\ 6 & 7 & 1 & 1 \\ 7 & 5 & 1 & 7 \end{bmatrix} - \begin{bmatrix} 7 & 1 & 2 & 1 \\ 6 & 3 & 0 & 2 \\ 7 & 0 & 4 & 2 \end{bmatrix}$	$\begin{bmatrix} -0.707+1.707j & 0.2929-0.707j & -0.707-1.707j & -1.4142j \\ 0 & 1.4142-1.4142j & -0.2929+0.707j & 0.707-0.2929j \\ 0 & -1.707-0.707j & 1.707+0.707j & 0.707-1.707j \end{bmatrix}$
5	$\begin{bmatrix} 2 & 0 & 5 & 7 \\ 6 & 7 & 1 & 1 \\ 7 & 4 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 1 & 0 \\ 6 & 3 & 0 & 2 \\ 7 & 7 & 3 & 1 \end{bmatrix}$	$\begin{bmatrix} -0.707+1.707j & 0 & -1.4142-1.4142j & -0.2929-0.707j \\ 0 & 1.4142-1.4142j & -0.2929+0.707j & 0.707-0.2929j \\ 0 & -1.707+0.707j & 1.707-0.707j & -0.707-1.707j \end{bmatrix}$
6	$\begin{bmatrix} 2 & 6 & 3 & 5 \\ 6 & 5 & 7 & 7 \\ 7 & 3 & 7 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 7 & 0 & 7 \\ 6 & 1 & 6 & 0 \\ 7 & 6 & 2 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.707+1.707j & -0.707-0.2929j & -1.707+0.707j & -1.4142 \\ 0 & -1.4142-1.4142j & 0.707+0.2929j & -0.2929-0.707j \\ 0 & -0.707+1.707j & 0.707-1.707j & -1.707-0.707j \end{bmatrix}$

Second, let $h = 1/2$. Functions $g_k(t)$ and modulated symbols $B_{k,n}$ remain the same as in Table I. Table III shows the recalculated matrices $\mathbf{V}_k - \hat{\mathbf{V}}_k$, and $(e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$. We can see that all matrices have full rank. Hence, this pair of codewords achieves full diversity and also satisfies rank criterion proposed by Zhang and Fitz [4].

Finally, for $h = 1/4$ we can similarly verified that full diversity is achieved.

To show that codewords (25) and (26) with modulation indices $h \notin \{1/2, 1/4, 1/8\}$ may not achieve full spatial diversity, we choose these codewords with modulation index

TABLE III

RANK VERIFICATION FOR FULL-RESPONSE CPM $M = 8, h = 1/2$

k	0	1	2
$V_k - V_k^{'}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
$e^{j2\pi h V_k} - e^{j2\pi h V_k^{'}}$	$\begin{bmatrix} 2 & 2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{bmatrix}$

3	4	5	6
$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{bmatrix}$

TABLE IV

RANK VERIFICATION FOR FULL-RESPONSE CPM $M = 8, h = 1/3$

k	0	1	2
$V_k - V_k^{'}$	$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix}$
$e^{j2\pi h V_k} - e^{j2\pi h V_k^{'}}$	$\begin{bmatrix} -1.732j & -1.732j & 1.732j & 0 \\ 0 & -1.732j & -1.5+0.866j & 1.732j \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.732j & -1.5-0.866j & 0 & 1.5+0.866j \\ 0 & -1.732j & -1.5+0.866j & 1.732j \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.732j & 1.5+0.866j & -1.5-0.866j & 0 \\ 0 & 1.5+0.866j & -1.732j & -1.5-0.866j \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3	4	5	6
$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$
$\begin{bmatrix} -1.732j & 1.5-0.866j & 0 & -1.5+0.866j \\ 0 & 1.5+0.866j & -1.732j & -1.5-0.866j \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.732j & -1.5-0.866j & 0 & 1.5+0.866j \\ 0 & -1.5+0.866j & 1.5+0.866j & 1.5-0.866j \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.732j & 0 & -1.5+0.866j & 1.5-0.866j \\ 0 & -1.5+0.866j & 1.5+0.866j & 1.5-0.866j \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1.732j & 1.5-0.866j & 0 & -1.5+0.866j \\ 0 & -0.732j & -1.5+0.866j & 1.732j \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$h = 1/3$. Table IV shows the recalculated matrices $\mathbf{V}_k$, $\hat{\mathbf{V}}_k$, and $\Delta\mathbf{V}_k = (e^{\mathbf{V}_k} - e^{\hat{\mathbf{V}}_k})$. Observe that all matrices $\Delta\mathbf{V}_k$ do not have full rank. Hence, this pair of codewords does not achieve full diversity.

VI. Conclusion

Space-time code design for CPM is more difficult than for linear modulation due to modulator nonlinearity. Computer search for CPM space-time codes is difficult, so general code construction rules that guarantee full spatial diversity for ST-CPM systems are useful. By using a linear decomposition of CPM signals, we have identified the rank criterion for M-ary full-response CPM that determines the set of allowable modulation indices. Several examples show that full spatial diversity is achieved for ST-CPM systems that meet the established rank criterion.

Disclaimer

The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U. S. Government.

VII. Appendix

A. Proof of Lemma 1

Proof: Denote $d = (a + b)_{\mathrm{mod}\ M}$. Note that $d \in \{0, 1, \cdots, M-1\}$ and that d can be presented in the radix-2 form $d = \sum_{i=0}^{k-1} 2^i \psi_i(d)$. Then

$$\begin{aligned}(a+b)_{\mathrm{mod}\ M} &= d \Leftrightarrow \\ \sum_{i=0}^{k-1} 2^i \psi_i(d) &= \left(\sum_{i=0}^{k-1} 2^i [\psi_i(a) + \psi_i(b)] \right)_{\mathrm{mod}\ M}. \quad (27)\end{aligned}$$

From (27) we note that $\psi_i(d) = (\psi_i(a) + \psi_i(b) + c_{i-1})_{\mathrm{mod}\ 2}$. Then it follows $(a+b)_{\mathrm{mod}\ M} = \sum_{i=0}^{k-1} 2^i (\psi_i(a) + \psi_i(b) + c_{i-1})_{\mathrm{mod}\ 2}$. If we note that the carry bit c_{-1} into the addition of the lowest bits is always zero, we obtain our lemma claim. ⊡

B. Proof of Lemma 2

Proof: $\mathbf{B} = f(\mathbf{A})$ is equivalent to $b_{i,j} = f(a_{i,j}) = (\sum_{m=0}^{j-k} a_{i,m})_{\mathrm{mod}\ M}$, for $\forall b_{i,j} \in \mathbf{B}$. From Lemma 1, it follows

$$\begin{aligned}b_{i,j} &= \left[\sum_{m=0}^{j-w} \psi_0(a_{i,m}) \right]_{\mathrm{mod}\ 2} + \\ &\sum_{l=1}^{k-1} 2^l \left[\sum_{m=0}^{j-w} \psi_l(a_{i,m}) + c_{l-1} \right]_{\mathrm{mod}\ 2}. \quad (28)\end{aligned}$$

On the other hand, $b_{i,j}$ can be written in the radix-2 form

$$b_{i,j} = \sum_{l=0}^{k-1} 2^i \psi_l(b_{i,j}). \quad (29)$$

From equations (28) and (29) it follows

$$\psi_0(b_{i,j}) = \left[\sum_{m=0}^{j-w} \psi_0(a_{i,m}) \right]_{\mathrm{mod}\ 2} = g(\psi_0(a_{i,j})). \quad (30)$$

Since equation (30) holds for $\forall b_{i,j} \in \mathbf{B}$ and $\forall a_{i,j} \in \mathbf{A}$, we can generalize our conclusion: $\mathbf{B} = f(\mathbf{A})$ implies $\Phi_0(\mathbf{B}) = g(\Phi_0(\mathbf{A}))$, what was our claim. ⊡

C. Proof of Lemma 3

Proof: We will prove this lemma by contradiction. Assume that the matrix $\mathbf{A}$ does not have full rank. Then, there exists at least one $K_l \neq 0$, for $l = \{1, \cdots, r\}$, such that

$$\left[\sum_{l=1}^{r} K_l a_{l,j} \right]_{\mathrm{mod}\ M} = \left[\sum_{l=1}^{r} K_l \sum_{i=0}^{k-1} 2^i \psi_i(a_{l,j}) \right]_{\mathrm{mod}\ M} = 0, \quad (31)$$

where $j = \{1, \cdots, n\}$. Using Lemma 1, (31) can be written as $[\sum_{l=1}^{r} K_l \psi_0(a_{l,j})]_{\mathrm{mod}\ 2} + \cdots + \left[2^{k-1} \sum_{l=1}^{r} K_l (\psi_{k-1}(a_{l,j}) + C_{k-2})\right]_{\mathrm{mod}\ 2} = 0$. From the fact that $\Phi_0(\mathbf{A})$ has full rank, it follows that $[\sum_{l=1}^{r} K_l \psi_0(a_{l,j})]_{\mathrm{mod}\ 2} \neq 0$ for $\forall j = \{1, \cdots, n\}$, and $[\sum_{l=1}^{r} K_l \psi_0(a_{l,j})]_{\mathrm{mod}\ 2} \neq \left[\sum_{l=1}^{r} K_l \sum_{m=1}^{k-1} 2^m (\psi_m(a_{l,j}) + C_{m-1})\right]_{\mathrm{mod}\ 2}$, which leads to a contradiction. ⊡

D. Proof of Lemma 5

Proof: We will prove this lemma by contradiction. Assume $\Delta \mathbf{s}(t)$ as defined in (20) does not have full rank. Then there exists some nonzero complex vector $[k_1, k_2, \cdots, k_{L_t}]^T$ that satisfies

$$\begin{aligned}&\sum_{n=0}^{N_c-1} \left[\sum_{i=1}^{L_t} k_i (\beta_{0,n}^{(i)} - \hat{\beta}_{0,n}^{(i)}) \right] g_0(t - nT_c) + \cdots + \quad (32)\\ &\sum_{n=0}^{N_c-1} \left[\sum_{i=1}^{L_t} k_i (\beta_{R-1,n}^{(i)} - \hat{\beta}_{R-1,n}^{(i)})) \right] g_{R-1}(t - nT_c) = 0,\end{aligned}$$

where $\beta_{k,n}^{(i)} - \hat{\beta}_{k,n}^{(i)} = \exp(j2\pi h v_{k,n}^{(i)}) - \exp(j2\pi h \hat{v}_{k,n}^{(i)})$. During the first symbol interval $0 \leq t \leq T_c$, $g_k(t - iT_c) = 0$ for $i = 1, 2, \ldots, N_c - 1$. For $0 \leq t \leq T_c$ equation (32) can be modified to $\left[\sum_{i=1}^{L_t} k_i (\beta_{0,0}^{(i)} - \hat{\beta}_{0,0}^{(i)})\right] g_0(t) + \cdots + \left[\sum_{i=0}^{L_t} k_i (\beta_{R-1,0}^{(i)} - \hat{\beta}_{R-1,0}^{(i)})\right] g_{R-1}(t) = 0$. Note from (12) that every function $g_k(t)$ has a different combination of functions $c_0^{(l)}(t + m_l T_c)$. It follows that all functions $g_k(t)$ are linearly independent over interval $0 \leq t \leq T_c$. We can conclude that $k_1 \left[\beta_{0,0}^{(1)} - \hat{\beta}_{0,0}^{(1)}\right] + \cdots + k_{L_t} \left[\beta_{0,0}^{(L_t)} - \hat{\beta}_{0,0}^{(L_t)}\right] = 0$. It can be shown by induction that for $n = 0, 1, \ldots, N_c - 1$ the following equation holds

$$k_1 \left[\beta_{0,n}^{(1)} - \hat{\beta}_{0,n}^{(1)}\right] + \cdots + k_{L_t} \left[\beta_{0,n}^{(L_t)} - \hat{\beta}_{0,n}^{(L_t)}\right] = 0. \quad (33)$$

Equation (33) implies that matrix $\Delta \mathbf{V} = (e^{\mathbf{V_k}} - e^{\hat{\mathbf{V}}_\mathbf{k}})$ does not have full rank, which contradicts our assumption. ⊡

References

[1] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. on Information Theory*, vol. 44, pp. 744–765, Mar. 1998.

[2] A. R. Hammons and H. E. Gamal, "On the theory of space-time codes for PSK modulation," *IEEE Trans. on Information Theory*, vol. 46, pp. 524–542, Mar. 2000.

[3] Y. Liu, M. P. Fitz and O. Y. Takeshita, "A rank criterion for QAM space-time codes," *IEEE Trans. on Information Theory*, vol. 48, pp. 3062–3079, Dec. 2002.

[4] X. Zhang and M. P. Fitz, "Space-time code design with continuous phase modulation," in *IEEE Journal on Selected Areas in Communications*, vol. 21, pp. 783–792, June 2003.

[5] X. Zhang and M. P. Fitz, "Space-time code design with CPM transmission," in *IEEE Int. Symp. on Inform. Theory*, (Washington, DC), p. 327, June 2001.

[6] Hsiao-feng Lu and P. V. Kumar, "Rate-diversity tradeoff of space-time codes with fixed alphabet and optimal constructuins for (PSK) modulation," *IEEE Trans. on Information Theory*, vol. 49, pp. 2747–2751, Oct. 2003.

[7] Hsiao-feng Lu and P. V. Kumar, "Constructing optimal space-time codes over various signal constellations ," *IEEE Globecom*, pp. 1973–1977, Nov. 2003.

[8] T. Aulin, N. Rydbeck and C. W. Sundberg, "Continuous phase modulation—part I and part II," *IEEE Trans. on Communications*, vol.COM 29, pp. 196–225, Mar. 1981.

[9] U. Mengali and M. Morelli, "Decomposition of M-ary CPM signals into PAM waveforms," *IEEE Trans. on Information Theory*, vol. 41, pp. 1265–1275, Sept. 1995.

[10] G. L. Stüber, *Principles of mobile communication 2e.* Kluwer, 2001.