ECE 2025

HW# 4 FALL 2000

Solution

4.1

(a) $x(t) = 2\cos(500\pi t) + 3\cos(500\pi t)\cdot\cos(150\pi t - 0.2\pi)$

$$x(t) = 2\cos(500\pi t) + 3\left\{\frac{1}{2}e^{j500\pi t} + \frac{1}{2}e^{-j500\pi t}\right\}\cdot\left\{\frac{1}{2}e^{-j0.2\pi}\cdot e^{j150\pi t} + \frac{1}{2}e^{j0.2\pi}e^{-j150\pi t}\right\}$$

$$\Rightarrow x(t) = 2\cos(500\pi t) + \frac{3}{4}e^{-j0.2\pi}e^{j650\pi t} + \frac{3}{4}e^{j0.2\pi}\cdot e^{-j650\pi t} + \frac{3}{4}e^{-j0.2\pi}e^{-j350\pi t} + \frac{3}{4}e^{j0.2\pi}e^{j350\pi t}$$

From above expression we conclude that there are three frequecy components: 350π, 500π, 650π $\frac{rad}{s}$.

Therefore, the fundamental frequency is $\omega_0 = 50\pi$ $\frac{rad}{s}$, which is the greatest common divisor of 350π, 500π and 650π.

(b)

Since $x(t)$ is given in terms of sinusoids, we use inverse Euler formula instead of computing the integral:

Using the result of part (a), we have:

$$x(t) = e^{j500\pi t} + e^{-j500\pi t} + \frac{3}{4}e^{-j0.2\pi}e^{j650\pi t} + \frac{3}{4}e^{j0.2\pi}e^{-j650\pi t} +$$
$$+ \frac{3}{4}e^{-j0.2\pi}e^{-j350\pi t} + \frac{3}{4}e^{j0.2\pi}e^{j350\pi t}$$

Since $\omega_0 = 50\pi$, we have:

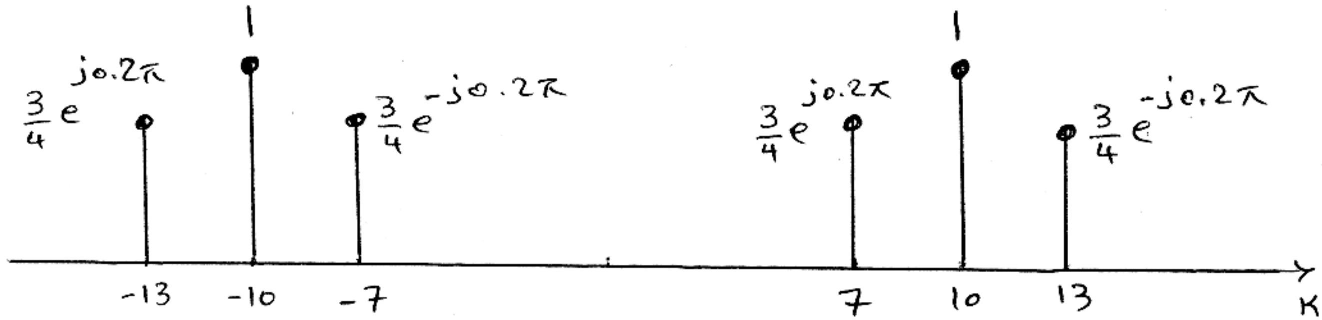
$350\pi = 7\omega_0 \rightarrow K=7$, $a_7 = \frac{3}{4}e^{j0.2\pi}$, $a_{-7} = \frac{3}{4}e^{-j0.2\pi}$

$500\pi = 10\omega_0 \rightarrow K=10$, $a_{10} = 1$, $a_{-10} = 1$

$650\pi = 13\omega_0 \rightarrow K=13$, $a_{13} = \frac{3}{4}e^{-j0.2\pi}$, $a_{-13} = \frac{3}{4}e^{j0.2\pi}$

$a_K = 0$ for all K except $K = \pm 7, \pm 10, \pm 13$

(c)



4.2

(a) Since $y(t) = 2x(t) + 3$, two signals $x(t)$ and $y(t)$ have the same fundamental frequency ω_0.

By definition we can write:

$$\sum_{k=-\infty}^{+\infty} b_k e^{jk\omega_0 t} = 2\left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\right) + 3$$

$$= (2a_0 + 3) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} (2a_k) e^{jk\omega_0 t}$$

Therefore:

$b_0 = 2a_0 + 3$

$b_k = 2a_k$ for all k except $k=0$
i.e., $k = \pm 1, \pm 2, \pm 3, \ldots$

(b) Note that since $z(t) = x(t-1)$, we conclude that $z(t)$ is also periodic with the same fundamental frequency as of $x(t)$:

Let $z(t) = \sum_{k=-\infty}^{+\infty} b_k e^{jk\omega_0 t}$ and

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Then we have: $\left(\text{using } z(t) = x(t-1)\right)$

$$\sum_{k=-\infty}^{+\infty} b_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o (t-1)}$$

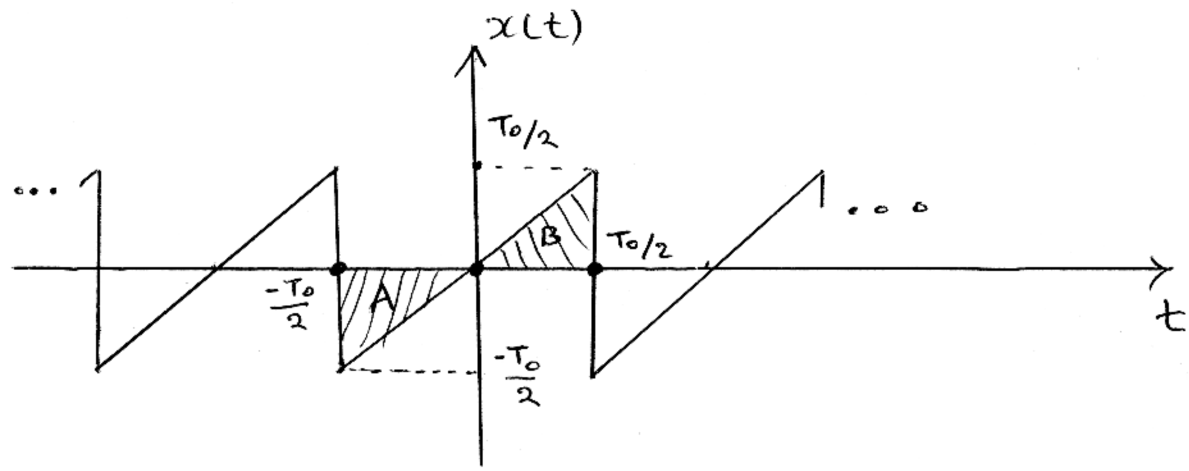
$$= \sum_{k=-\infty}^{+\infty} \left(a_k \cdot e^{-jk\omega_o}\right) e^{jk\omega_o t}$$

By equating the terms in the both sides:

$$b_k = a_k e^{-jk\omega_o} \qquad k = 0, \pm 1, \pm 2, \ldots$$

4.3

(a)



(b) $a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} t \cdot dt = 0$

Alternative method: $a_0 = \frac{1}{T_0}$ (the area under $x(t)$ in one Period)

$$\rightarrow \quad a_0 = \frac{1}{T_0}(A+B) \quad \text{where} \quad \begin{cases} A: \text{ area of } x(t) & -\frac{T_0}{2} \le t \le 0 \\ B: \text{ area of } x(t) & 0 \le t \le \frac{T_0}{2} \end{cases}$$

But $A = -B$

Thus $\quad a_0 = \frac{1}{T_0}(0) = 0$

(c)

for $k \neq 0 \qquad a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} t\, e^{-jk\omega_0 t} \cdot dt$

where $\quad \omega_0 = \frac{2\pi}{T_0}$

$$\rightarrow \quad a_k = \frac{1}{T_0}\left[\frac{t}{-jk\omega_0} e^{-jk\omega_0 t}\right]_{-T_0/2}^{T_0/2} + \frac{1}{jk\omega_0 T_0}\int_{-T_0/2}^{T_0/2} e^{-jk\omega_0 t} \cdot dt$$

$$\rightarrow \quad a_k = \frac{jT_0}{2k\pi}\cos(k\pi) - \underbrace{\frac{jT_0}{2k^2\pi^2}\sin k\pi}_{=0}$$

Thus $\quad a_k = \frac{jT_0}{2k\pi}\cos(k\pi) \qquad\qquad k \neq 0$

(d) In order to plot the spectrum of $x(t)$ we need to write $x(t)$ in terms of complex exponentials. Since Fourier series representation is in a complex exponential form, we can derive spectrum directly from the Fourier Series:

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} t\, e^{-jk\omega_0 t}\ dt$$

$$= \frac{t}{-jk\omega_0 T_0} e^{-jk\omega_0 t}\Bigg]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} - \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\left(\frac{1}{-jk\omega_0}\right)e^{-jk\omega_0 t}dt$$

$$= \frac{j}{2k\pi}\left(\frac{T_0}{2}e^{-jk\omega_0\frac{T_0}{2}} + \frac{T_0}{2}e^{jk\omega_0\frac{T_0}{2}}\right) - \frac{j}{2k\pi}\left[\frac{j}{-k\omega_0}e^{-jk\omega_0 t}\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}}$$

$$= \frac{jT_0}{2k\pi}\cos(k\pi) - \frac{T_0}{4\pi^2k^2}\left(e^{-jk\omega_0\frac{T_0}{2}} - e^{jk\omega_0\frac{T_0}{2}}\right)$$

$$= \frac{jT_0}{2k\pi}\cos(k\pi) + \frac{T_0}{4\pi^2k^2}\left(j2\sin(k\pi)\right)$$

$$= \frac{jT_0}{2k\pi}\cos(k\pi)$$

or...

$$= \frac{jT_0}{2k\pi}(-1)^k$$

Therefore, the coefficients for k = -3, -2, -1, 0, 1, 2, 3 are as follows.

$$a_0 = 0$$

$$a_1 = -j\frac{2}{\pi} = \frac{2e^{-j\frac{\pi}{2}}}{\pi}$$

$$a_{-1} = j\frac{2}{\pi} = \frac{2e^{j\frac{\pi}{2}}}{\pi}$$

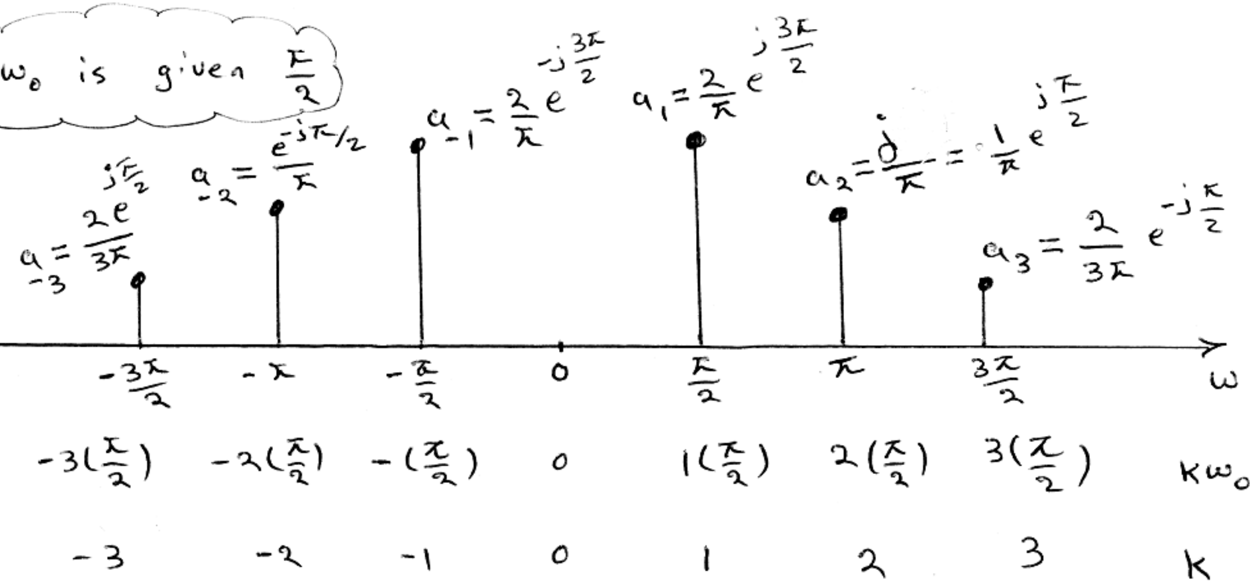
$$a_2 = \frac{j}{\pi} = \frac{e^{j\frac{\pi}{2}}}{\pi}$$

$$a_{-2} = \frac{-j}{\pi} = \frac{e^{-j\frac{\pi}{2}}}{\pi}$$

$$a_3 = -j\frac{4}{6\pi} = \frac{2e^{-j\frac{\pi}{2}}}{3\pi}$$

$$a_{-3} = j\frac{4}{6\pi} = \frac{2e^{j\frac{\pi}{2}}}{3\pi}$$

ω_0 is given $\frac{\pi}{2}$

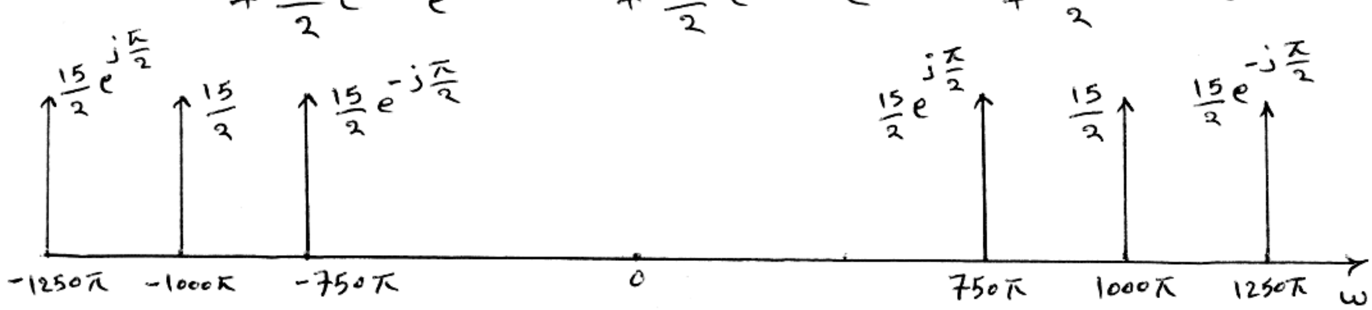


4.4

(a) $x(t) = 15\cos(1000\pi t) + 30\sin(250\pi t)\cos(1000\pi t)$

$$\rightarrow x(t) = 15\cos(1000\pi t) + 30\left[\frac{1}{2j}e^{j250\pi t} - \frac{1}{2j}e^{-j250\pi t}\right]\cdot\left[\frac{1}{2}e^{j1000\pi t} + \frac{1}{2}e^{-j1000\pi t}\right]$$

$$\rightarrow x(t) = \underbrace{\frac{15}{2}e^{j1000\pi t} + \frac{15}{2}e^{-j1000\pi t}} + \frac{15}{2}e^{-j\frac{\pi}{2}}e^{j1250\pi t} + \frac{15}{2}e^{j\frac{\pi}{2}}e^{-j1250\pi t} + \frac{15}{2}e^{j\frac{\pi}{2}}e^{j750\pi t} + \frac{15}{2}e^{-j\frac{\pi}{2}}e^{-j750\pi t}$$



(b) Since $\underbrace{\omega_0 = 250\pi}_{\text{is the largest number which}}$ divides 750π, 1000π, 1250π,

we conclude that $\omega_0 = 250\pi$ is the fundamental frequency (rad/s). Thus the signal is periodic with period $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{250\pi} = 8$ msec

(c) $f_s \geqslant 2 f_{max}$ $\quad f_{max} = \frac{1250\pi}{2\pi}$

$\rightarrow f_s \geqslant 1250$ HZ $\quad$ Thus $(f_s)_{min} = 1250$ HZ

4.5
===

Since $x[n]$ is given in sinusoids form, we suggest $x_1(t) = A_1 \cos(2\pi f_1 t + \varphi_1)$ in which the parameters A_1, f_1, and φ_1 have to be found.

we use $x[n] = x_1(nT_s)$ where $T_s = \frac{1}{2000}$

$$\rightarrow 4\cos(0.125\pi n + \frac{\pi}{8}) = A_1 \cos(2\pi f_1 \frac{n}{2000} + \varphi_1)$$

$$\Rightarrow A_1 = 4 \;,\; \varphi_1 = \frac{\pi}{8} \;,\; f_1 = 125 \text{ HZ}$$

Therefore, $x_1(t) = 4\cos(2\pi(125)t + \frac{\pi}{8})$

To find $x_2(t)$ (that would give the same $x[n]$),

we use the fact that adding $2\pi k$ ($k = \pm1, \pm2, \ldots$)

to $\hat{\omega}$ (the frequency of $x[n]$) does not change

anything. i.e., $x[n] = 4\cos((0.125\pi + 2k\pi)n + \frac{\pi}{8})$

If we start with $x_2(t) = A_2\cos(2\pi f_2 t + \varphi_2)$,

then $x_2(t)$ can be found from $x_2(nT_s) = x[n]$

Following the same approach we had for $x_1(t)$,

we obtain:

$$A_2 = 4 \quad , \quad \varphi_2 = \frac{\pi}{8} \quad ,$$

$$\frac{2\pi f_2}{2000} = 0.125\pi + 2k\pi \qquad k = \pm1, \pm2, \ldots$$

$$\rightarrow f_2 = 125 + 2000k \qquad k = \pm1, \pm2, \ldots$$

Since we require $f_2 < 2000$, thus we choose

$k = -1 \rightarrow f_2 = -1875$

$$\rightarrow x_2(t) = 4\cos(-2\pi(1875)t + \frac{\pi}{8})$$

$$= 4\cos(2\pi(1875)t - \frac{\pi}{8})$$

because $\cos(-\theta) = \cos(\theta)$ for any θ

(b) First we find $x(t)$ from the spectrum:

$$x(t) = 10\cos(2\pi(30)t + \frac{3\pi}{8}) + 8\cos(2\pi(120)t + \frac{\pi}{3})$$

Now, we obtain $x[n]$. Since $f_s = 120$ Samples/sec we expect that there is an aliasing term introduced by the second cosine term in $x(t)$.

$x[n] = x(nT_s)$ where $T_s = \frac{1}{120}$

$$x[n] = 10\cos(\frac{\pi}{2}n + \frac{3\pi}{8}) + 8\cos(2\pi n + \frac{\pi}{3})$$

Thus $x[n]$ has two frequency components:

$\hat{\omega}_1 = \frac{\pi}{2}$, $\hat{\omega}_2 = 2\pi$

To plot the spectrum of $x[n]$, we treat it similar to the plot of the spectrum for continuous-time signals. But we need to keep in mind two things:

(1) We plot the spectrum of any arbitrary $x[n]$ in the interval $-\pi \leq \hat{\omega} \leq \pi$.

(2) For those frequency components that are not in the interval $-\pi \leq \omega \leq \pi$, we subtract (or add) multiples of (2π) such

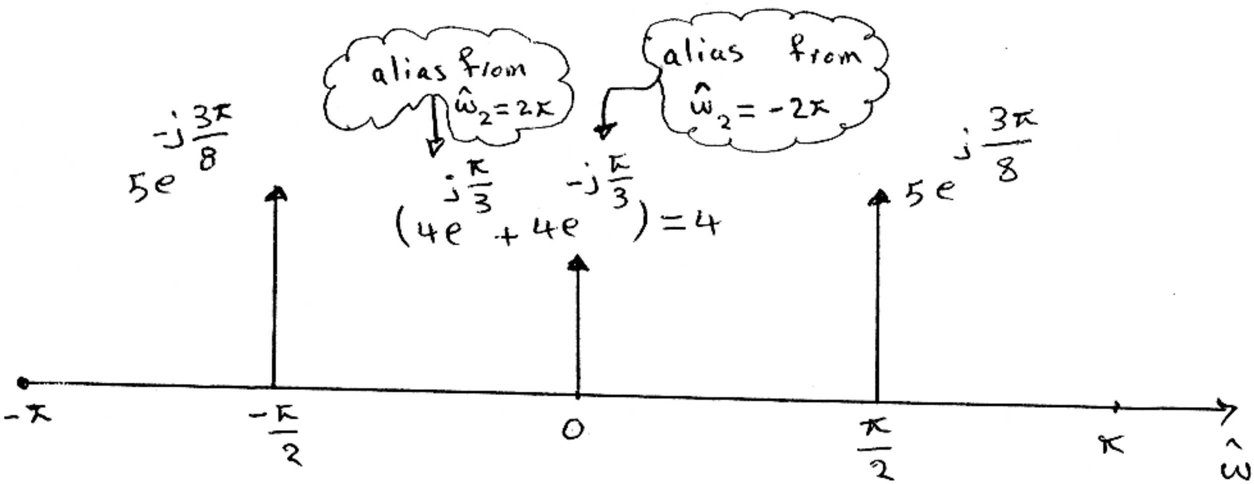
that the new frequency lies in the interval

$-\pi \le \hat{\omega} \le \pi$.

Note that step (2) in the above is required for all aliasing terms.

Now, since $\hat{\omega}_1 = \frac{\pi}{2}$, step (2) is not required.

But for $\hat{\omega}_2 = 2\pi$, we follow step (2). This gives a new $\hat{\omega}_2 = 2\pi - 2\pi = 0$



Note to the DC term introduced by aliasing frequecies.