ECE 2025: INTRODUCTION TO DIGITAL SIGNAL PROCESSING
Assignment #6 Solutions

PROBLEM 6.3:

$$y[n] = 2x[n] - 5x[n-1] + 2x[n-2]$$

a) Linearity test:

Let $x[n] = a_1x_1[n] + a_2x_2[n]$, then

$y[n] = 2(a_1x_1[n] + a_2x_2[n]) - 5(a_1x_1[n-1] + a_2x_2[n-1]) + 2(a_1x_1[n-2] + a_2x_2[n-2])$

$y[n] = 2a_1x_1[n] + 2a_2x_2[n] - 5a_1x_1[n-1] - 5a_2x_2[n-1] + 2a_1x_1[n-2] + 2a_2x_2[n-2]$

$y[n] = 2a_1x_1[n] - 5a_1x_1[n-1] + 2a_1x_1[n-2] + 2a_2x_2[n] - 5a_2x_2[n-1] + 2a_2x_2[n-2]$

$y[n] = a_1(2x_1[n] - 5x_1[n-1] + 2x_1[n-2]) + a_2(2x_2[n] - 5x_2[n-1] + 2x_2[n-2])$

$y[n] = a_1y_1[n] + a_2y_2[n]$

Yes, $y[n]$ is linear.

Time-invariance test:

We have to show that a delay in $x[n]$ by n_0, $x[n-n_0]$, results in a delay in $y[n]$ to $y[n-n_0]$. To do this, we start by letting $w[n]$ be the output when the input $x[n]$ is delayed by n_0 giving

$$w[n] = 2x[n-n_0] - 5x[n-1-n_0] + 2x[n-2-n_0]$$

Then if we delay $y[n]$ by n_0, we obtain

$$y[n-n_0] = 2x[n-n_0] - 5x[n-1-n_0] + 2x[n-2-n_0]$$

We can see that

$$y[n-n_0] = w[n]$$

so $y[n]$ is a time-invariant system.

Causal:

$y[n]$ is causal since it depends only on $x[n]$, $x[n-1]$ and $x[n-2]$ which are in the present or in the past.

b) Since $y[n]$ is an LTI system, the frequency response is of the form

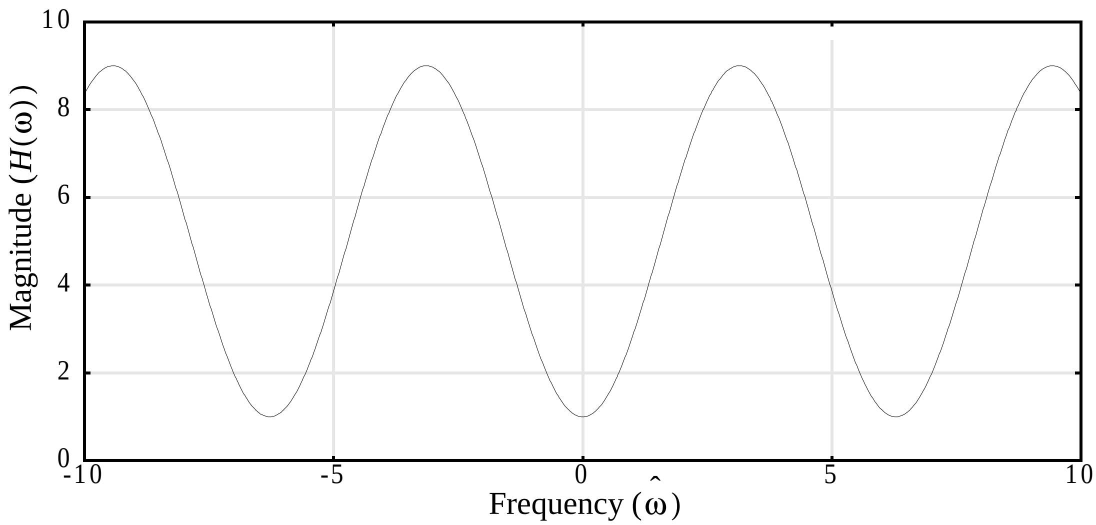
$$H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

For $y[n]$, $M = 2$ and $b_k = \{2, -5, 2\}$. Therefore,

$$H(\hat{\omega}) = 2e^{-j\hat{\omega}0} - 5e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} = (2e^{j\hat{\omega}} - 5 + 2e^{-j\hat{\omega}})e^{-j\hat{\omega}} = (4\cos(\hat{\omega}) - 5)e^{-j\hat{\omega}}$$

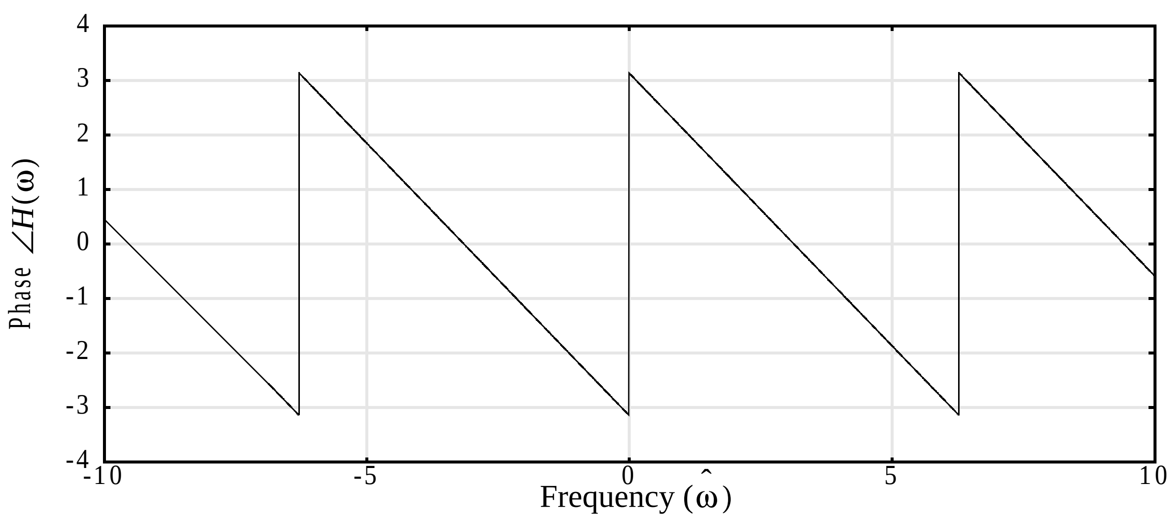
c) The figure below shows a plot for the magnitude of the frequency response

$$|H(\hat{\omega})| = |(4\cos(\hat{\omega}) - 5)e^{-j\hat{\omega}}| = |4\cos(\hat{\omega}) - 5|$$



The figure below shows a plot of the phase in the range of $-\pi < \theta < \pi$ for the frequency response (note the $+\pi$ because $4\cos(\hat{\omega}) - 5$ is always negative)

$$\angle H(\hat{\omega}) = \angle(4\cos(\hat{\omega}) - 5)e^{-j\hat{\omega}} = -(\hat{\omega} + \pi)$$



d) Given the input $x_2[n] = 4 + 4\cos(0.5\pi(n-1))$, $y_1[n]$ can be calculated as follows. First calculate $H(\hat{\omega})$ for the two frequencies in $x_2[n]$.

$$H(0) = (4\cos(0) - 5)e^{-j0} = -1$$
$$H(\pi/2) = (4\cos(\pi/2) - 5)e^{-j(\pi/2)} = -5e^{-j(\pi/2)} = 5e^{j(\pi/2)}$$

With these two frequency response values, $y_1[n]$ can be determined as follows

$$y_1[n] = (H(0))(4) + |H(\pi/2)||4|\cos(0.5\pi(n-1) + \arg(H(\pi/2)))$$
$$y_1[n] = -4 + 20\cos(0.5\pi(n-1) + \pi/2) = -4 + 20\cos(0.5\pi n)$$

PROBLEM 6.4:

$$y[n] = (x[n-1])^2$$

a) Linearity test:

Let $x[n] = a_1x_1[n] + a_2x_2[n]$, then

$$y[n] = (a_1x_1[n-1] + a_2x_2[n-1])^2 \neq (a_1x_1[n-1])^2 + (a_2x_2[n-1])^2$$

so $y[n]$ is not linear.

Time-invariance test:

We have to show that a delay in $x[n]$ by n_0, $x[n-n_0]$, results in a delay in $y[n]$ to $y[n-n_0]$. To do this, we start by letting $w[n]$ be the output when the input $x[n]$ is delayed by n_0 giving

$$w[n] = (x[n-1-n_0])^2$$

Then if we delay $y[n]$ by n_0, we obtain

$$y[n-n_0] = (x[n-n_0-1])^2$$

We can see that

$$y[n-n_0] = w[n]$$

so $y[n]$ is a time-invariant system.

Causal: $y[n]$ is causal since it depends only on $x[n-1]$ which is in the past.

b) $y_1[n] = (e^{j0.75\pi(n-1)} + e^{-j0.75\pi(n-1)})^2 = e^{j1.5\pi(n-1)} + e^{-j1.5\pi(n-1)} + 2e^{j0}$

$$= 2\cos(1.5\pi(n-1)) + 2$$

Notice that this sampled sinusoid has a frequency outside of the range $[-\pi, \pi]$ that a D-to-C converter assumes for signal reconstruction. This signal will therefore result in the folded alias as follows

$$y_1[n] = 2\cos((-0.5)\pi(n-1)) + 2 = 2\cos(0.5\pi(n-1)) + 2$$

instead of the desired signal.

PROBLEM 6.5:

a) The impluse response for $y_1[n]$ is $h_1[n] = \delta[n] - \delta[n-1]$ with coefficients $b_k = \{1, -1\}$. The coefficients for $h_2[n]$ are $b_k = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$. The impulse response for the overall cascade system can be tabulated as follows.

n	0	1	2	3	4	5	6	7	8	9	10	11
$h_2[n]$	1	1	1	1	1	1	1	1	1	1		
$h_1[n]$	1	-1										
	1	1	1	1	1	1	1	1	1	1		
		-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	
$h[n]$	1	0	0	0	0	0	0	0	0	0	-1	

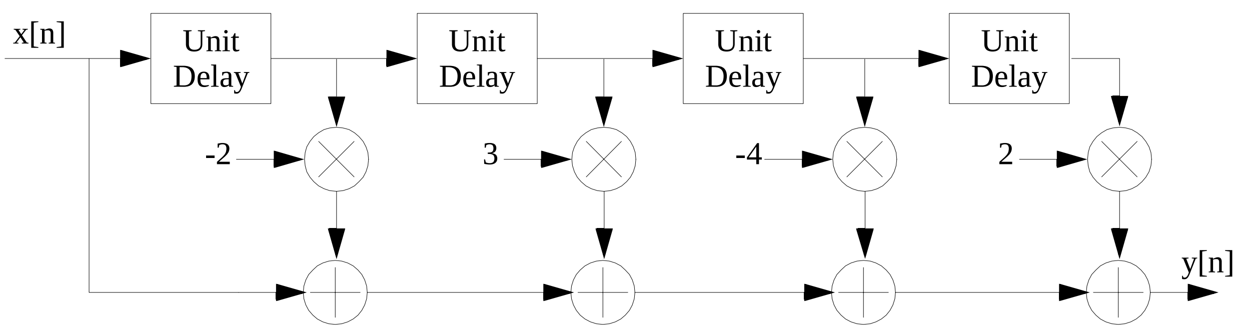
Therefore, $h[n] = \delta[n] - \delta[n-10]$.

b) Given the impulse response $h[n]$ from a), then $y[n] = x[n] - x[n-10]$.

PROBLEM 6.6:

$$y[n] = x[n] - 2x[n-1] + 3x[n-2] - 4x[n-3] + 2x[n-4]$$

a) The block diagram for $y[n]$ is as follows.



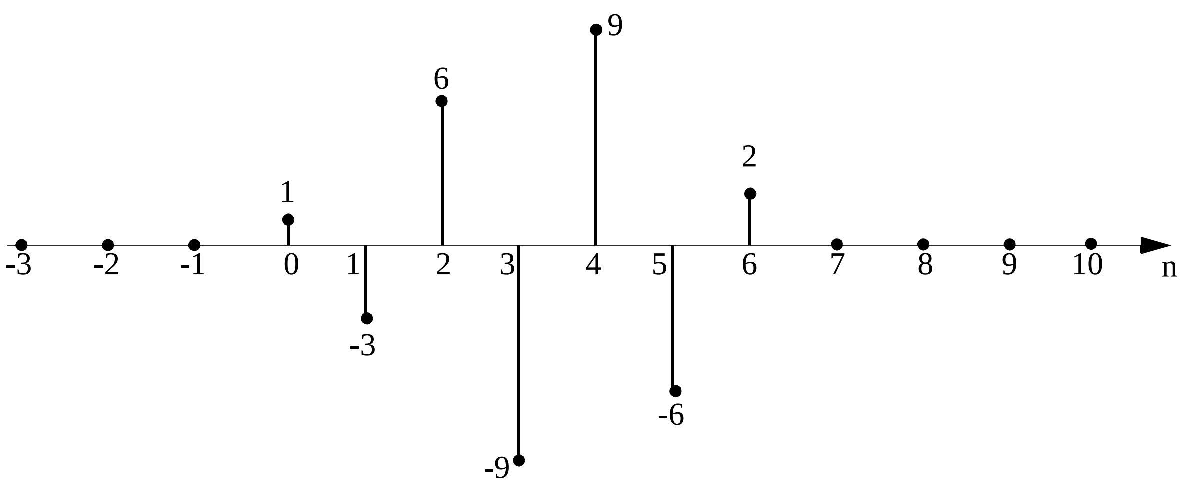
b) The impulse response for $y[n]$ can be found by using $x[n] = \delta[n]$ which results in

$$y[n] = h[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-2] - 4\delta[n-3] + 2\delta[n-4]$$

c) $y[n]$ can be tabulated as follows.

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
$h[n]$				1	-2	3	-4	2						
$x[n]$				1	-1	1								
				1	-2	3	-4	2						
					-1	2	-3	4	-2					
						1	-2	3	-4	2				
$y[n]$				1	-3	6	-9	9	-6	2				

Plotting $y[n]$ gives



PROBLEM 6.7:

a) First, expand out $H(\hat{\omega})$

$$H(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{-j\pi/4}e^{-j\hat{\omega}})(1 - e^{j\pi/4}e^{-j\hat{\omega}})$$

$$H(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{j\pi/4}e^{-j\hat{\omega}} - e^{-j\pi/4}e^{-j\hat{\omega}} + e^{-j2\hat{\omega}})$$

$$H(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{-j\hat{\omega}}(e^{j\pi/4} + e^{-j\pi/4}) + e^{-j2\hat{\omega}})$$

$$H(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - \sqrt{2}e^{-j\hat{\omega}} + e^{-j2\hat{\omega}})$$

$$H(\hat{\omega}) = 1 - \sqrt{2}e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j\hat{\omega}} - \sqrt{2}e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}}$$

$$H(\hat{\omega}) = 1 + (1 - \sqrt{2})e^{-j\hat{\omega}} + (1 - \sqrt{2})e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}}$$

Considering the form $H(\hat{\omega}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$, the coefficients are $b_k = \{1, 1 - \sqrt{2}, 1 - \sqrt{2}, 1\}$

with $M = 3$. From these coefficients, $y[n]$ can be written as

$$y[n] = \sum_{k=0}^{3} b_k x[n-k] = x[n] + (1-\sqrt{2})x[n-1] + (1-\sqrt{2})x[n-2] + x[n-3] .$$

b) If the input is $x[n] = \delta[n]$, then the output is

$$y[n] = \delta[n] + (1-\sqrt{2})\delta[n-1] + (1-\sqrt{2})\delta[n-2] + \delta[n-3]$$

c) With an input of $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, the output will be of the form $y[n] = H(\hat{\omega})(Ae^{j\phi}e^{j\hat{\omega}n})$. Therefore, $y[n] = 0$ for all n only when $H(\hat{\omega}) = 0$. By consider the roots of $H(\hat{\omega})$, we know that $H(\hat{\omega}) = 0$ when: i) $(1 + e^{-j\hat{\omega}}) = 0$, ii) $(1 - e^{-j\pi/4}e^{-j\hat{\omega}}) = 0$ or iii) $(1 - e^{j\pi/4}e^{-j\hat{\omega}}) = 0$. Solving each of these gives

i) $e^{-j\hat{\omega}} = -1$ which is true when $\hat{\omega} = \pi + 2\pi l$

ii) $e^{-j\hat{\omega}} = e^{j\pi/4}$ which is true when $\hat{\omega} = -\pi/4 + 2\pi l$

iii) $e^{-j\hat{\omega}} = e^{-j\pi/4}$ which is true when $\hat{\omega} = \pi/4 + 2\pi l$

Therefore, $y[n] = 0$ for all n when $\hat{\omega} = \{\pi + 2\pi l, \pi/4 + 2\pi l, -\pi/4 + 2\pi l\}$, l an integer.

For the specific range of $-\pi \le \hat{\omega} \le \pi$, the possible $\hat{\omega}$ are $-\pi$, π, $\pi/4$, and $-\pi/4$, but, note that π and $-\pi$ result in the same frequency.