GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Spring 2001
Problem Set #4

Assigned: 26-January-01
Due Date: Week of 5-February-01

Quiz #1 will be held in lecture on Friday 2-February-01. It will cover material from Chapters 2 and 3, as represented in Problem Sets #1, #2, and #3.
Closed book, calculators permitted, and one hand-written formula sheet ($8\frac{1}{2}'' \times 11''$, both sides) CHECK OUT PREVIOUS EXAMS ON WEBCT.

Reading: In *DSP First*, all of Chapter 3 on *Spectrum Representation*, especially pp. 48–73. Also read the Fourier Series Notes on WebCT.

$\Longrightarrow$ **Please check the "Bulletin Board" often. All official course announcements are posted there.**

ALL of the **STARRED** problems will have to be turned in for grading. A solution will be posted to the web. Some problems have solutions similar to those found on the CD-ROM.

Your homework is always due in recitation at the beginning of class. After the beginning of your assigned recitation time, the homework is considered late and will be given a zero. **However, this week, the Monday and Tuesday Recitation sections may turn in this homework assignment with Lab #3 at your Lab time on Wednesday and Thursday respectively. Wednesday and Thursday Recitations turn in this assignment at your normal recitation time.**

PROBLEM 4.1*:

Let $x(t)$ be the signal

$$x(t) = [10 + 5\cos(2000\pi t + \pi/5)]\cos(10000\pi t).$$

(a) Use Euler's relation to expand $x(t)$ as a sum of complex exponential signals and show that it can be expressed in the Fourier series form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

(b) Determine the fundamental frequency ω_0 of this signal.

(c) What is the "DC value" of this signal?

(d) Determine all of the non-zero coefficients a_k of this signal and plot the spectrum of this signal. **Note carefully that you should be able to do this without evaluating any integrals.**

PROBLEM 4.2*:

A periodic signal $x(t)$ is described over one period $0 \le t \le T_0$ by the equation

$$x(t) = \begin{cases} \dfrac{2t}{T_0} & 0 \le t < T_0/2 \\ 0 & T_0/2 \le t \le T_0. \end{cases}$$

We have seen that such a periodic signal can be represented by the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{where} \quad a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt$$

(a) Sketch the periodic function $x(t)$ for $-T_0 < t < 2T_0$.

(b) Determine a_0, the D.C. coefficient for the Fourier series.

(c) Set up the Fourier *analysis* integral for determining a_k for $k \neq 0$. (Insert proper limits and integrand.) What integration technique from calculus could you apply to aid in evaluating this integral?

You do not have to evaluate the integral in part (c). If you are curious, the answer is:

$$a_k = \begin{cases} \dfrac{(1+j\pi k)e^{-j\pi k} - 1}{2\pi^2 k^2} = \dfrac{(1+j\pi k)(-1)^k - 1}{2\pi^2 k^2} & k \neq 0 \\ \text{your value of } a_0 \text{ found in (b)} & k = 0 \end{cases}$$

Note: A similar problem is worked out in great detail in Section 33.5.3 of the Fourier Series Notes on WebCT.

PROBLEM 4.3*:

A periodic signal $x(t)$ with a period $T_0 = 10$ is described *over one period,* $0 \leq t \leq 10$, by the equation

$$x(t) = \begin{cases} 0 & 0 \leq t \leq 5 \\ 2 & 5 < t \leq 10. \end{cases}$$

This signal can be represented by the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

which is valid for all time $-\infty < t < \infty$.

(a) Sketch the periodic function $x(t)$ for $-10 < t < 20$.

(b) Determine a_0, the D.C. coefficient of the Fourier Series.

(c) Use the Fourier *analysis* integral [1] (for $k \neq 0$)

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

to find the first ($k = 1$) Fourier series coefficient, a_1. Note: $\omega_0 = 2\pi / T_0$.

(d) If we add a constant value of one to $x(t)$, we obtain the signal $y(t) = 1 + x(t)$ with $y(t)$ given over one period by

$$y(t) = \begin{cases} 1 & 0 \leq t \leq 5 \\ 3 & 5 < t \leq 10. \end{cases}$$

This signal can also be represented by a Fourier series,

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$

Explain how b_0 and b_1 are related to a_0 and a_1. (Note: You should not have to evaluate any new integrals explicitly to answer this question.)

[1] The Fourier integral can be done over any period of the signal; in this case, the most convenient choice is from -5 to 0.

PROBLEM 4.4*:

We have seen that a periodic signal $x(t)$ can be represented by the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{1}$$

It turns out that we can transform many operations on the signal into corresponding operations on the Fourier coefficients a_k. For example, suppose that we want to consider a new periodic signal $y(t) = \frac{dx(t)}{dt}$. What would the Fourier coefficients be for $y(t)$? To see this, we simply need to differentiate the Fourier series representation; i.e.,

$$y(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right] = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt}\left[e^{jk\omega_0 t}\right] = \sum_{k=-\infty}^{\infty} a_k \left[(jk\omega_0)e^{jk\omega_0 t}\right]. \tag{2}$$

Thus, we see that $y(t)$ is also in the Fourier series form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}, \quad \text{where} \quad b_k = (jk\omega_0)a_k$$

but in this case the Fourier series coefficients are related to the Fourier series coefficients of $x(t)$ by $b_k = (jk\omega_0)a_k$. This is a nice result because it allows us to find the Fourier coefficients *without* actually doing the differentiation of $x(t)$ and *without* doing any tedious evaluation of integrals to obtain the Fourier coefficients b_k. It is a *general* result that holds for every periodic signal and its derivative.

We can use this style of manipulation to obtain some other useful results for Fourier series. In each case below, use Equation (1) as the starting point and the given definition for $y(t)$ to express $y(t)$ as a Fourier series and then manipulate the equation so that you can pick off an expression for the Fourier coefficients b_k as a function of the original coefficients a_k.

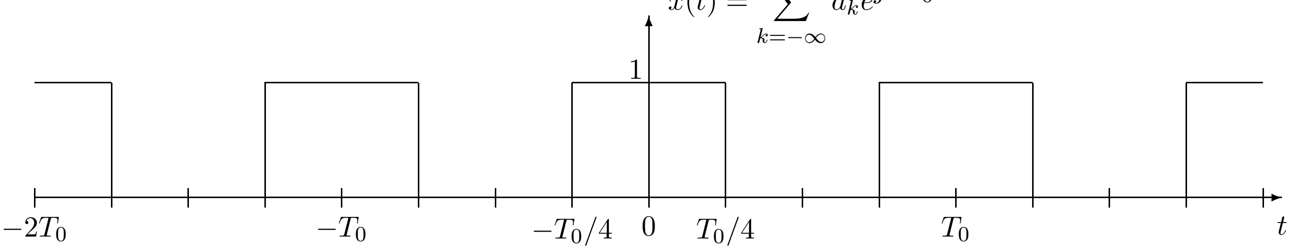
(a) Suppose that $y(t) = Ax(t)$ where A is a real number; i.e., $y(t)$ is just a scaled version of $x(t)$. Show that the Fourier coefficients for $y(t)$ are $b_k = Aa_k$.

(b) Suppose that $y(t) = x(t - t_d)$ where t_d is a real number; i.e., $y(t)$ is just a delayed version of $x(t)$. Show that the Fourier coefficients for $y(t)$ in this case are $b_k = a_k e^{-jk\omega_0 t_d}$.

PROBLEM 4.5*:

Consider the periodic function $x(t)$ plotted below.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$



(a) Find the "DC" value a_0 and the other Fourier coefficients a_k for $k \neq 0$ in the Fourier series representation of $x(t)$.

(b) Sketch the waveform of the signal $y(t) = 2x(t - T_0/2)$ and use the results of Problem 4.4 to write down the Fourier series coefficients b_0 and b_k for $k \neq 0$ for the periodic signal $y(t)$ without evaluating any integrals. **Note: You will use this result in Section 4 of Lab #3.**