EE-2025 Fall-2001

Lecture 23
H(z) & Frequency Response
30-Nov-01

Final Exam Info

- Calendar: Final Exam(s)
  - Period 3, Monday, 10-Dec @ 2:50 pm
    - NOON Lecture
    - **Must take exam with your assigned section**
  - Period 11, Thurs, 13-Dec @ 11:30am
    - 11 AM Lecture
- **Report CONFLICTS immediately !!!!**
  - e.g., 3 exams in one day
- Reviews will be held on Sunday & Wednesday
  - 6:00 PM in ECE Auditorium

FSK Challenge Contest

- Advanced FSK Demodulator & Decoder
  - Encrypted Messages & Clues
  - Degraded Signals
    - Modem is out of spec: frequencies are not quite right, bit rate is approximate, interference added
- Prize:
  - Final Exam Exemption(s)
- Details will be posted by Saturday morning
  - Help Session: Sat @ noon in ECE Aud.

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 263-279
- Other Reading:
  - Recitation: Ch. 8, pp. 261-272
    - POLES & ZEROS

LECTURE OBJECTIVES

- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS

Use H(z) to get Freq. Response

$$z = e^{j\hat{\omega}}$$

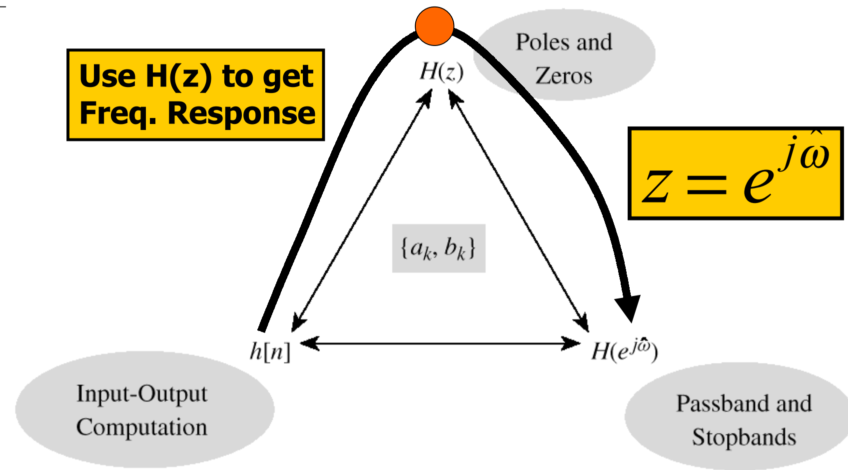


Figure 8.13 Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k, b_k\}$ play a central role.

H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

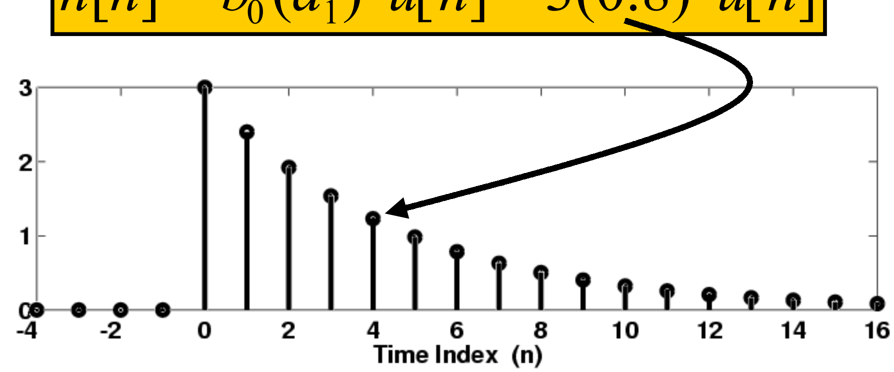
$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



Derivation of H(z)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

First-Order Transform Pair

- GEOMETRIC SEQUENCE:

$$h[n] = b a^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}$$

- USE KNOWN TRANSFORM PAIR:

$$h[n] = b a^n u[n] = 3(0.8)^n u[n]$$

$$H(z) = \sum_{n} 3(0.8)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} 3(0.8)^n z^{-n} = \frac{3}{1 - 0.8 z^{-1}}$$

DELAY PROPERTY of X(z)

- DELAY in TIME<-->Multiply X(z) by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

$$\text{Proof:} \quad \sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+1)}$$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} = z^{-1} X(z)$$

Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION H(z)
  - Use **DELAY** PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

- **READ** the FILTER COEFFS:

H(z)

$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8 z^{-1}} \right) X(z)$$

POLES & ZEROS

- ROOTS of NUMERATOR & DENOMINATOR

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: H(z) -> inf

EXAMPLE: Poles & Zeros

- VALUE of H(z) at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8 z^{-1}}$$

$$H(-1) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

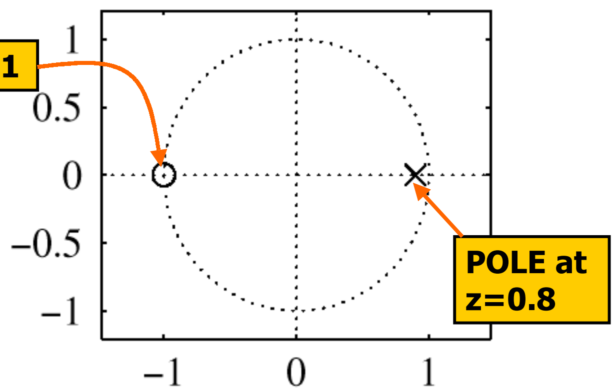
ZERO at z= -1

$$H(\tfrac{4}{5}) = \frac{2 + 2(\tfrac{4}{5})}{1 - 0.8(\tfrac{4}{5})} = \frac{\tfrac{7}{5}}{0} \rightarrow \infty$$

POLE at z=0.8

POLE-ZERO PLOT

ZERO at z = -1



$$\frac{2+2z^{-1}}{1-0.8z^{-1}}$$

FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

FREQ. RESPONSE FORMULA

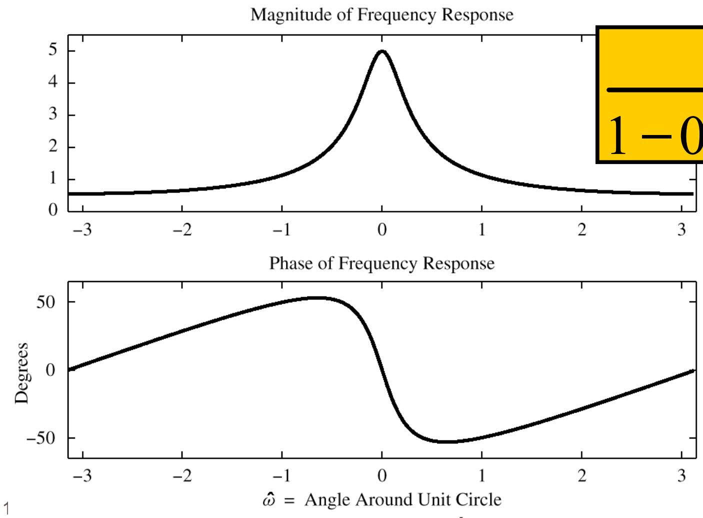
$$H(z) = \frac{1}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{1}{1-0.8e^{-j\hat{\omega}}}$$

$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{1}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{1}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{1}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{1}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{1}{1.64-1.6\cos\hat{\omega}}$$

$$@\hat{\omega}=0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{1}{0.04} = 25 \qquad @\hat{\omega}=\pi\,?$$

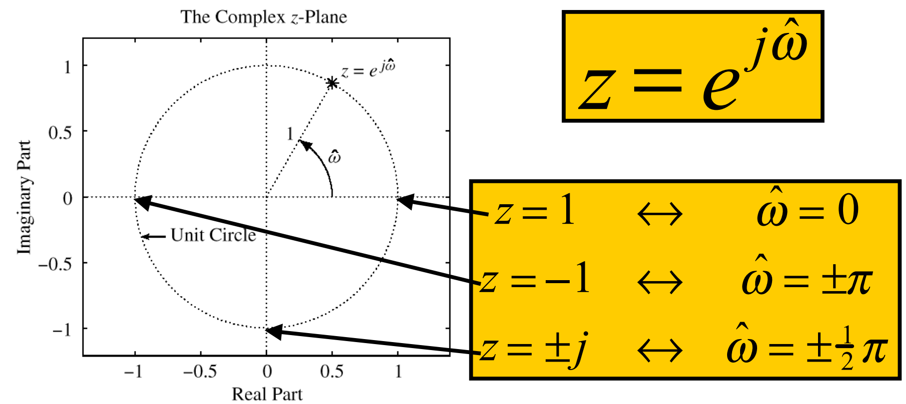
FREQ. RESPONSE from H(z)



$$\frac{1}{1-0.8e^{-j\hat{\omega}}}$$

UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



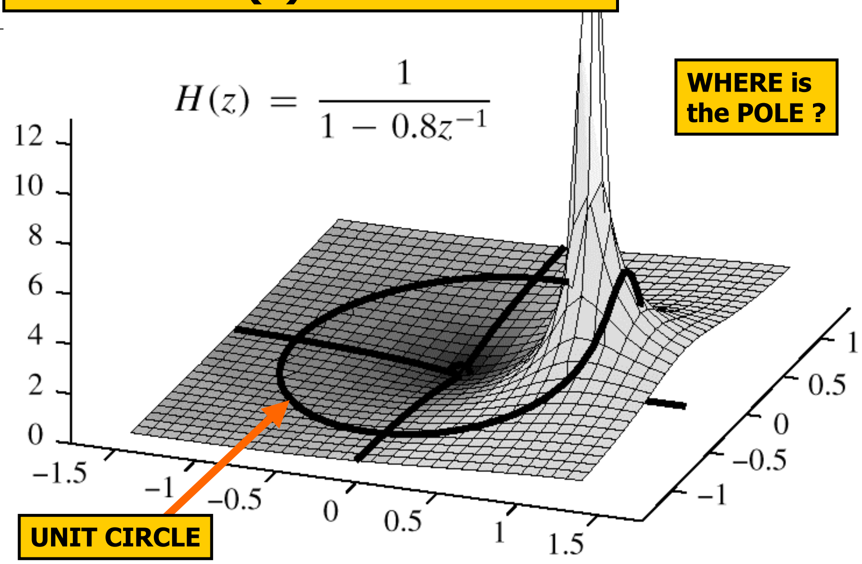
$$z = e^{j\hat{\omega}}$$

$z = 1 \leftrightarrow \hat{\omega} = 0$

$z = -1 \leftrightarrow \hat{\omega} = \pm\pi$

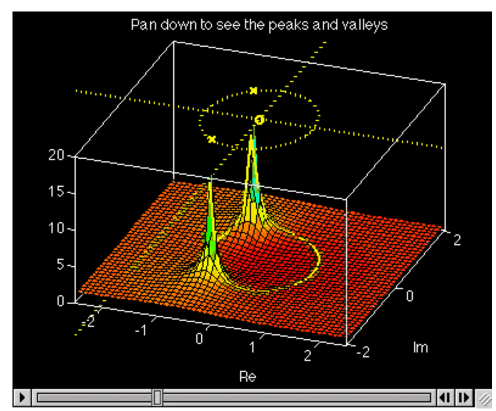
$z = \pm j \leftrightarrow \hat{\omega} = \pm\frac{1}{2}\pi$

3-D VIEWPOINT: EVALUTE H(z) EVERYWHERE

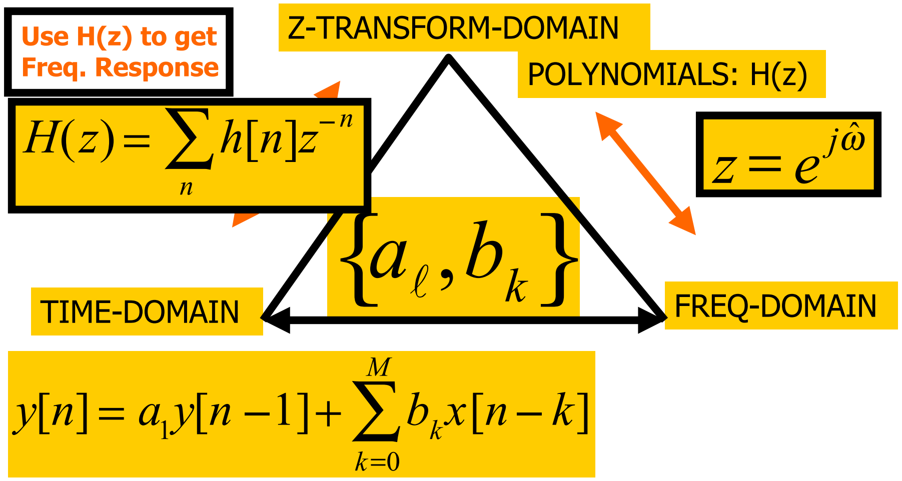


MOVIE for H(z) in 3-D

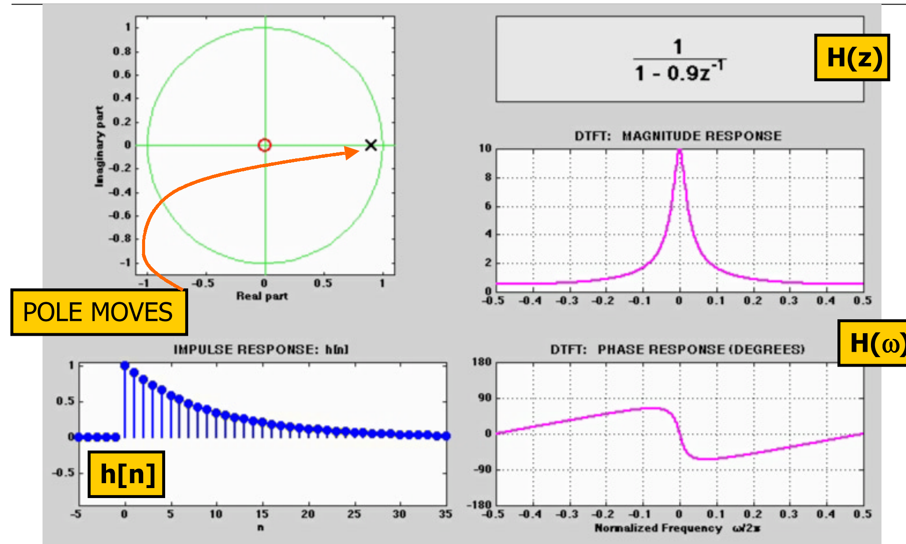
- POLES to H(z) to Frequency Reponse
  - TWO POLES SHOWN



THREE DOMAINS



3 DOMAINS MOVIE: IIR



POP QUIZ

- Given: $H(z)=\dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find the Impulse Response, h[n]
- Find the output, y[n]
  - When $x[n]=\cos(0.25\pi n)$

POP QUIZ: Invert Z

- Given:

$$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}=\frac{2}{1-0.8z^{-1}}+\frac{2z^{-1}}{1-0.8z^{-1}}$$

- Find the Impulse Response, h[n]
  - Use the DELAY PROPERTY

$$h[n]=2(0.8)^n u[n]+2(0.8)^{n-1}u[n-1]$$

SINUSOIDAL RESPONSE

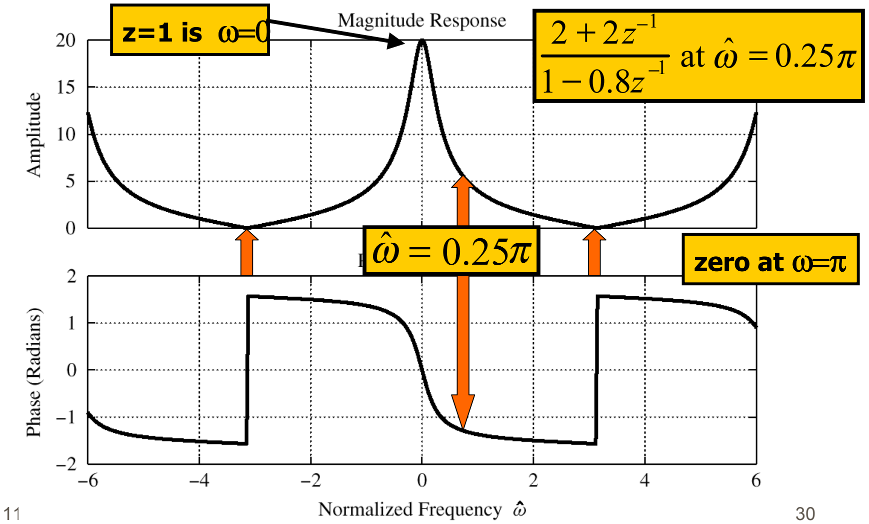
- x[n] = SINUSOID => y[n] is SINUSOID
- Get MAGNITUDE & PHASE from H(z)

$$\text{if } x[n] = e^{j\hat{\omega}n}, \text{ then}$$

$$y[n] = \mathcal{H}(\hat{\omega})e^{j\hat{\omega}n}$$

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

- Given: $H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find output, y[n], when $x[n] = \cos(0.25\pi n)$
  - Evaluate at $z = e^{j0.25\pi}$

$$H(z) = \frac{2+2(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182\cos(0.25\pi n - 0.417\pi)$$