GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 2003
Problem Set #4

Assigned: 5-Sept-03
Due Date: Week of 14-Sept-03

Reading: In *Signal Processing First*, sections of Chapter 3 on *Fourier Series*.

Warning: This is one of the more mathematically intense homeworks of the semester. Start early, and if you are finding the math challenging, don't panic. Most of the other homeworks are not this hard.

Quiz 1 will be held during lecture on 19-Sept-03.

$\Longrightarrow$ Please check the "Bulletin Board" often. All official course announcements are posted there.

ALL of the **STARRED** problems will have to be turned in for grading. A solution will be posted to the web.

Your homework is due in recitation at the beginning of class. After the beginning of your assigned recitation time, the homework is considered late and will be given a zero.

PROBLEM 4.1*:

Suppose we have a radar system fixed at the origin $(0, 0)$. It picks up a signal reflected by a target at position $p(t) = (x_0, y_0 + vt)$, where (x_0, y_0) is the position of the target at $t = 0$, v is the velocity of the target, and the target is flying parallel to the y-axis.

The signal received by a particular airborne radar system is given by:

$$s(t) = \cos\left(2\pi f_c t - 2\pi f_c \frac{2}{c}\sqrt{x_0^2 + (y_0 + vt)^2}\right),$$

where f_c the frequency (in Hertz) of a continuous-wave sinusoidal signal $\cos(2\pi f_c t)$ that is broadcast by the radar and c is the speed of light. Just leave as x_0, y_0, ω_c, c, and v symbolic constants in this problem; don't substitute specific numbers in for any of them. Note that $\sqrt{x_0^2 + (y_0 + vt)^2}$ measures the distance from the target to the radar. Dividing by c converts distance (in meters) to time (in seconds); the "2" appears since the signal has to travel from the radar to the target and back again.

(a) Using the ideas on p. 61 of the textbook, find the instantaneous frequency $f_i(t)$ (in Hertz) of the signal given by $s(t)$. (We're using "s" to represent the signal instead of "x," as we usually do, to avoid confusion with the position "x.")

(b) What is $f_i(0)$ if $y_0 = 0$?

PROBLEM 4.2*:

Use the Fourier series analysis integral to compute the Fourier series coefficients a_k for the periodic function $x(t)$, with fundamental period T_0, where a single period is specified by $x(t) = \exp(-a|t|)$ for $-T_0/2 \leq 0 \leq T_0/2$. (Be sure to notice the absolute value.)

The best way to handle the absolute value is to break the integral into two parts; one part for $-T_0/2 \leq t \leq 0$ and another part for $0 \leq t \leq T_0/2$. (Yes, the calculus and algebra can get tedious, but it's good for you to slog through one of these kinds of problems at least once.)

Simplify your answer as much as you can. It won't simplify very much, but there are some tricks you can and should use:

- You will come up with terms with denominators like $j\omega_0 k - a$ and $j\omega_0 k + a$. It's best to put everything over a single common denominator like $\omega_0^2 k^2 + a^2$.
- You will be able to get your answer in a form where you can use two inverse Euler's formulas; one will give you $\sin(\omega_0 kT/2)$, and another will give you $\cos(\omega_0 kT/2)$.

Hint: Since $x(t)$ is symmetric, i.e. $x(t) = x(-t)$, we know that the Fourier coefficients must consist solely of a real component without an imaginary component. If you're thinking in terms of polar coordinates, the phases must all be either 0 or π. That gives you a quick way to check to see if your answer makes sense.

PROBLEM 4.3*:

Which of the following signals are periodic? For those that are periodic, find the *fundamental period*, T_0, and the non-zero Fourier series coefficients a_k for all k. (Important: you should be able to do this problem *without* actually having to compute any integrals! While working the problem, stop yourself the moment you write an integral sign and spend some time thinking. You can generally rewrite the expressions using Euler's formula and expand things out.)

(a) $x(t) = 7 + \sin(1999\pi t - \pi/2) + 2\cos(2000\pi t + \pi/3)$

(b) $x(t) = \cos(\sqrt{3}\pi t + \pi/2) + \cos(\pi t - 4\pi/7)$

(c) $x(t)$ is given by the infinite sum

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{|k|^3 + 2} e^{j60k\pi t}$$

PROBLEM 4.4*:

Is the following signal periodic? If so, find the *fundamental period*, T_0, and the non-zero Fourier series coefficients a_k for all k. (As in the last problem, you should be able to do this problem without doing a Fourier integral.)

$$x(t) = 4\cos(250\pi t + \pi/3)\cos(1000\pi t)$$

PROBLEM 4.5:

Try Problem 4.4 from Homework 4 from the Fall 2001 offering of ECE2025. You can find it on the “WORD” section of WebCT. This will introduce you to the mindset you need to solve the next problem.

PROBLEM 4.6*:

Suppose $x(t)$, a periodic signal with fundamental period T_0, has Fourier series coefficients a_k, and $a_0 = 0$ (i.e., there is no D.C. component.) Let $y(t)$ be a periodic signal, specified on $0 \leq t \leq T_0$ by the integral

$$y(t) = \int_0^t x(\tau) d\tau. \tag{1}$$

When you take ECE2040, you will learn how to construct physical systems that are close approximations to such *integrators* using a resistors and capacitors.

(a) In this subpart, we'll find b_k, the Fourier coefficients of $y(t)$, in terms of a_k. This is a "property" of Fourier series that lies at a higher level of abstraction than the specific computation of Fourier series coefficients for particular signals, like you did in Problems 4.2, 4.3, and 4.4.

(i) Substitute the Fourier series representation of $x(\tau)$ into (1), exchange the order of integration and summation, and simplify the integral. You should get something that looks like this:

$$\sum_{k=-\infty}^{\infty} \frac{a_k}{\text{stuff}} e^{jk\omega_0 t} - \sum_{k=-\infty}^{\infty} \frac{a_k}{\text{stuff}}, \tag{2}$$

where by convention, we'll suppose that any term with an a_0 in it is zero.

Notice the second term doesn't have a t anywhere in it; hence, it must correspond to b_0. The second term gives us

$$b_0 = -\sum_{k=-\infty}^{\infty} \frac{a_k}{\text{stuff}} = \int_0^{T_0} y(t) dt.$$

(ii) By looking at the first term of *your* version of Equation 2 (in other words, we don't want to see the word "stuff" in your answer), find b_k for $k \neq 0$ in terms of a_k.

(iii) Why were we so careful to demand that $a_0 = 0$ in part (a)?

(b) Properties such as the "integration property of Fourier series," which you proved in part (a), are useful if you want to derive Fourier series coeffients for some function if you already have the Fourier series coefficients for some other function. This concept – using properties to massage some existing result into a new result you need – will appear over and over in ECE2025. (Come to think of it, it appears *everywhere* in math, science, and engineering.) Making this conceptual leap to thinking at more abstract levels will lead you to much success.

In this subpart, we'll crank through an example of the kind of mileage that you can get from a property. Consider the square wave in Chapter 3, Section 6.1 of the text. The Fourier coefficients of this square wave are given in Equation 3.32 of the text.

Let's make a minor modification; set $a_0 = 0$, so the square wave has a DC value of zero and we can apply your new integration property. Let's also take $T_0 = 4$.

(i) Sketch a carefully labeled plot of $x(t)$, the square wave from Chapter 3, Section 6.1 modified so that $a_0 = 0$.

(ii) Sketch a carefully labeled plot of $y(t)$.

Now think to yourself: Ah ha! The integral of my square wave is a triangle wave! I might want the Fourier coefficients of the triangle wave, but what a pain they would be to compute from scratch!

(iii) Use the property on the integration of Fourier series you discovered in part (a) to compute the Fourier series coefficients of the triangle wave $y(t)$.

Now think to yourself: That was a whole lot easier than computing the Fourier analysis integrals from scratch would have been!

(Hint: Clever folks could check their work by looking through Chapter 3.)

A side note to the mathematically inclined: Our proof in part (a) is not very rigorous; folks from from the math department might not be happy with us being so cavalier about switching the order of integration and summation. We're engineers, though, so it's good enough for us. It's possible to formulate a more solid proof using integration by parts, but that proof is complicated and beyond the scope of the things we want to do in ECE2025. If you have no idea what I'm talking about in this side note, don't worry about it; I just wanted to mention it in passing.