GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Spring 2004
Problem Set #12

Assigned: 10-April-04
Due Date: 23-April-04

This Homework can be turned at the last lecture on ***Friday, 23-April before Noon,*** or earlier that week.

Final Exam will be given on 30-April at 2:50 PM. One page ($8\frac{1}{2} \times 11''$) of **handwritten** notes allowed.

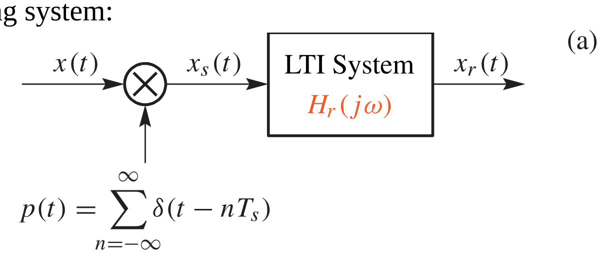
Reading: In *SP First*, Chapter 8: *IIR Filters*

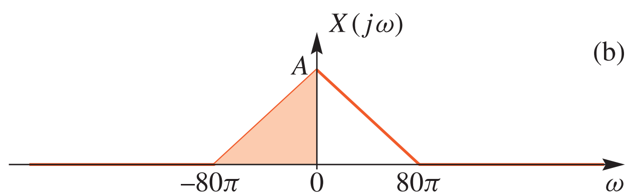
$\Longrightarrow$ **Please check the "Bulletin Board" often. All official course announcements are posted there.**

ALL of the **STARRED** problems will have to be turned in for grading. A solution will be posted to the web. Some problems have solutions similar to those found on the CD-ROM.

PROBLEM 12.1*:

The derivation of the Sampling Theorem involves the operations of impulse train sampling and reconstruction as shown in the following system:





A "typical" bandlimited Fourier transform of the input is also shown above.

(a) For the input with Fourier transform depicted above, determine the Nyquist rate, i.e., the smallest sampling rate $\omega_s = 2\pi/T_s$ so that $x_r(t) = x(t)$. Then plot $X_s(j\omega)$ for the value of $\omega_s = 2\pi/T_s$ that is equal to 1.5 times the Nyquist rate.

(b) If $\omega_s = 2\pi/T_s = 150\pi$ in the above system and $X(j\omega)$ is as depicted above, plot the Fourier transform $X_s(j\omega)$ and show that aliasing occurs. There will be an infinite number of shifted copies of $X(j\omega)$, so indicate the general pattern versus ω.

(c) For the conditions of part (b), i.e., $T_s = 1/75$, determine and sketch the Fourier transform of the output, $X_r(j\omega)$, if the frequency response of the LTI system is

$$H_r(j\omega) = \begin{cases} T_s & |\omega| \le \pi/T_s \\ 0 & |\omega| > \pi/T_s \end{cases}$$

PROBLEM 12.2*:

Signal Processing First, Chapter **8**, Problem **11**, page 240.

PROBLEM 12.3*:

Signal Processing First, Chapter **8**, Problem **13**, page 240–241.

PROBLEM 12.4*:

Signal Processing First, Chapter **8**, Problem **14**, page 241.
Note: There is an error in the text for problems **P-8.13** and **P-8.14**. The system $\mathcal{S}_6$ should be

$$\mathcal{S}_6 : \quad y[n] = \sum_{k=0}^{3} x[n-k] \qquad \text{(upper limit of 3, not 2)}$$

Copies of pages 240–241 (corrected) from the textbook are attached at the end of this document.

PROBLEM 12.5*:

Given a feedback filter defined via the recursion:

$$y[n] = 0.25y[n-2] + x[n] \qquad \text{(DIFFERENCE EQUATION)}$$

(a) Determine the impulse response $h[n]$, assuming the "at rest" initial condition.

(b) Determine the system function $H(z)$, and find the poles and zeros of the system.

(c) When the input to the system is the signal: $x[n] = (-1)^n u[n]$, determine the output signal $y[n]$, assuming the "at rest" initial condition (i.e., the output signal is zero for $n < 0$).
Hint: it should be possible to solve this problem with z-transforms; however, the algebra is easier if you do not factor the denominator of $H(z)$.

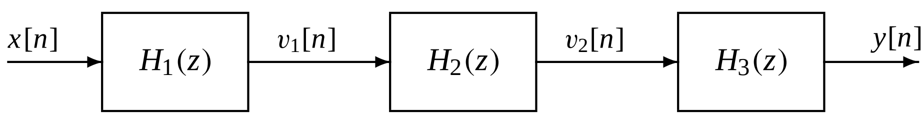
(d) Make a plot of the output signal $y[n]$ from part (c) over the range $-5 \leq n \leq 15$.

(e) Determine the region of the output $y[n]$ where the signal would be considered to have its *transient* behavior; likewise, identify the region where $y[n]$ has its *steady-state* behavior.

(f) Evaluate the frequency response at $\hat{\omega} = \pi$, and comment on the amplitude of the steady-state response signal found in part (e) versus $H(e^{j\pi})$.
Hint: for which value of z is $H(z)$ equal to $H(e^{j\pi})$?

PROBLEM 12.6:

In the following cascade of systems, all of the individual system functions, $H_i(z)$, are known.



$$H_1(z) = z^{-2} + z^{-3} \qquad H_2(z) = 6 - 3z^{-1} \qquad H_3(z) = \frac{2}{8 - 10z^{-1} + 3z^{-2}}$$

(a) Determine $H(z)$ the z-transform of the cascaded system. Simplify $H(z)$ by cancelling common factors in the numerator and denominator.

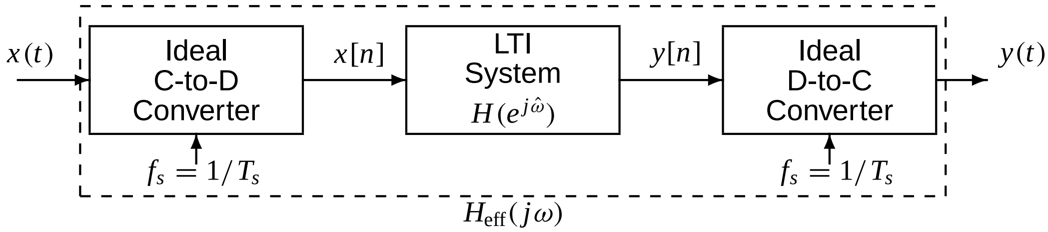
(b) Consider the impulse response of the cascaded system, i.e., the response $y[n]$ when the input is $x[n] = \delta[n]$. Prove that the impulse response has the form $h[n] = G\,\alpha^n$ for $n \geq 3$. Find values for α and G.

(c) Write *one difference equation* that defines the overall system in terms of $x[n]$ and $y[n]$ only.

PROBLEM 12.7:

This type of problem has often appeared on the Final Exam.

Consider the following system for discrete-time filtering of a continuous-time signal:

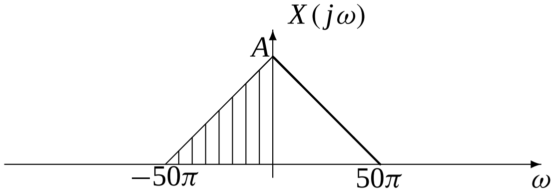


(a) Suppose that the discrete-time system is defined by the difference equation

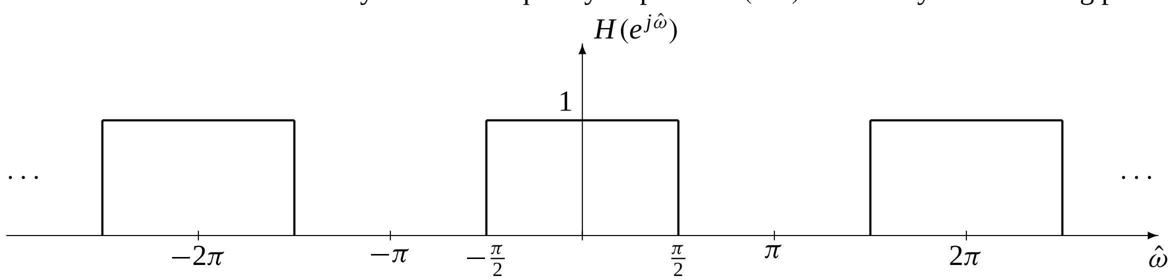
$$y[n] = 0.8y[n-1] + x[n] + x[n-2],$$

and the sampling rate of the input is $f_s = 200$ samples/second. Determine an expression for $H_{\text{eff}}(j\omega)$, the overall effective frequency response (versus analog frequency ω) of the above system. Use this result to find the output $y(t)$ when the input to the overall system is $x(t) = 2\cos(100\pi t)$.

(b) Assume that the input signal $x(t)$ has a bandlimited Fourier transform $X(j\omega)$ as depicted below. For this input signal, what is the *smallest* value of the sampling frequency f_s such that the Fourier transforms of the input and output satisfy the relation $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$?



(c) Assume that the discrete-time system has frequency response $H(e^{j\hat{\omega}})$ defined by the following plot:



Now, if $f_s = 200$ samples/sec, make a carefully labeled plot of $H_{\text{eff}}(j\omega)$, the effective frequency response of the overall system. Also plot $Y(j\omega)$, the Fourier transform of the output $y(t)$, when the input has Fourier transform $X(j\omega)$ as depicted in the graph of part (b).

(d) For the input in part (b) and the system in part (c), what is the smallest sampling rate such that the input signal passes through the lowpass filter unaltered; i.e., what is the minimum f_s such that $Y(j\omega) = X(j\omega)$?

(a) $H_a(z) = \dfrac{1 - z^{-1}}{1 + 0.77z^{-1}}$

(b) $H_b(z) = \dfrac{1 + 0.8z^{-1}}{1 - 0.9z^{-1}}$

(c) $H_c(z) = \dfrac{z^{-2}}{1 - 0.9z^{-1}}$

(d) $H_d(z) = 1 - z^{-1} + 2z^{-3} - 3z^{-4}$

P-8.12 Determine the inverse z-transform of the following:

(a) $X_a(z) = \dfrac{1 - z^{-1}}{1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$

(b) $X_b(z) = \dfrac{1 + z^{-2}}{1 + 0.9z^{-1} + 0.81z^{-2}}$

(c) $X_c(z) = \dfrac{1 + z^{-1}}{1 - 0.1z^{-1} - 0.72z^{-2}}$

P-8.13 For each of the pole-zero plots in Fig. P-8.13, determine which of the following systems (specified by either an $H(z)$ or a difference equation) matches the pole-zero plot.

$\mathcal{S}_1: \quad y[n] = 0.9y[n-1] + \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$

$\mathcal{S}_2: \quad y[n] = -0.9y[n-1] + 9x[n] + 10x[n-1]$

$\mathcal{S}_3: \quad H(z) = \dfrac{\frac{1}{2}(1 - z^{-1})}{1 + 0.9z^{-1}}$

$\mathcal{S}_4: \quad y[n] = \frac{1}{4}x[n] + x[n-1] + \frac{3}{2}x[n-2] + x[n-3] + \frac{1}{4}x[n-4]$

$\mathcal{S}_5: \quad H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4}$

$\mathcal{S}_6: \quad y[n] = \sum_{k=0}^{3} x[n-k]$

$\mathcal{S}_7: \quad y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5]$

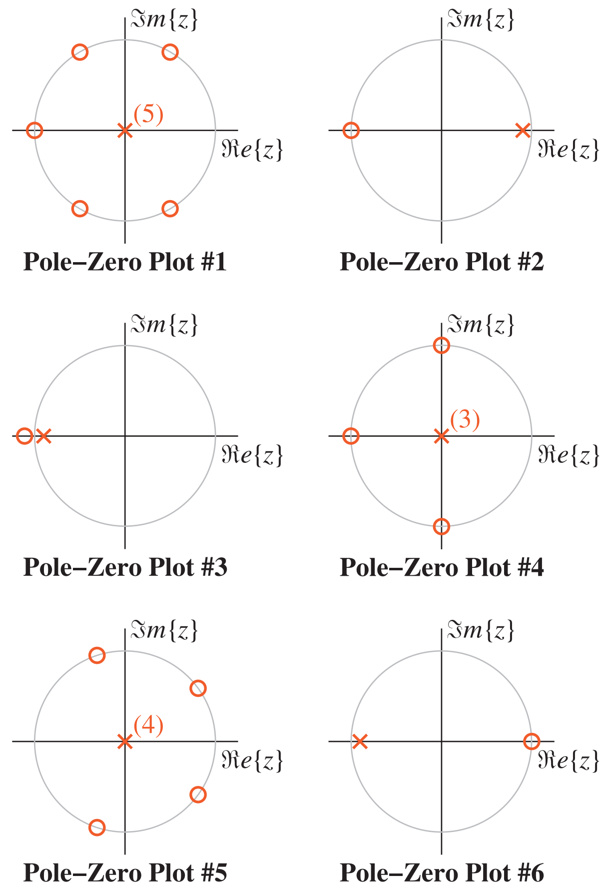


Figure P-8.13

P-8.14 For each of the frequency-response plots (A–F) in Fig. P-8.14, determine which of the following systems (specified by either an $H(z)$ or a difference equation) matches the frequency response.

Note: The frequency axis for each plot extends over the range $-\pi \le \hat{\omega} \le \pi$.

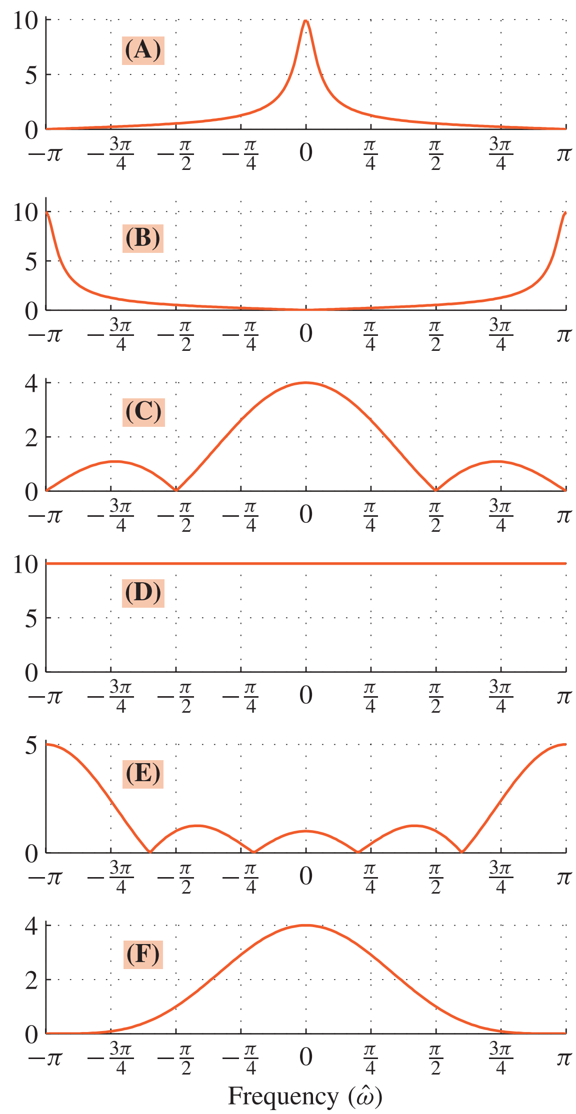


Figure P-8.14

$$\mathcal{S}_1: \quad y[n] = 0.9y[n-1] + \tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-1]$$

$$\mathcal{S}_2: \quad y[n] = -0.9y[n-1] + 9x[n] + 10x[n-1]$$

$$\mathcal{S}_3: \quad H(z) = \frac{\frac{1}{2}(1-z^{-1})}{1+0.9z^{-1}}$$

$$\mathcal{S}_4: \quad y[n] = \tfrac{1}{4}x[n] + x[n-1] + \tfrac{3}{2}x[n-2] + x[n-3] + \tfrac{1}{4}x[n-4]$$

$$\mathcal{S}_5: \quad H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4}$$

$$\mathcal{S}_6: \quad y[n] = \sum_{k=0}^{3} x[n-k]$$

$$\mathcal{S}_7: \quad y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5]$$

P-8.15 Given an IIR filter defined by the difference equation

$$y[n] = \tfrac{1}{2}y[n-1] + x[n]$$

(a) When the input to the system is a unit-step sequence, $u[n]$, determine the functional form for the output signal $y[n]$. Use the inverse z-transform method. Assume that the output signal $y[n]$ is zero for $n < 0$.

(b) Find the output when $x[n]$ is a complex exponential that starts at $n = 0$:

$$x[n] = e^{j(\pi/4)n}u[n]$$

(c) From (b), identify the steady-state component of the response, and compare its magnitude and phase to the frequency response at $\hat{\omega} = \pi/4$.