Model of Axial Loading 1

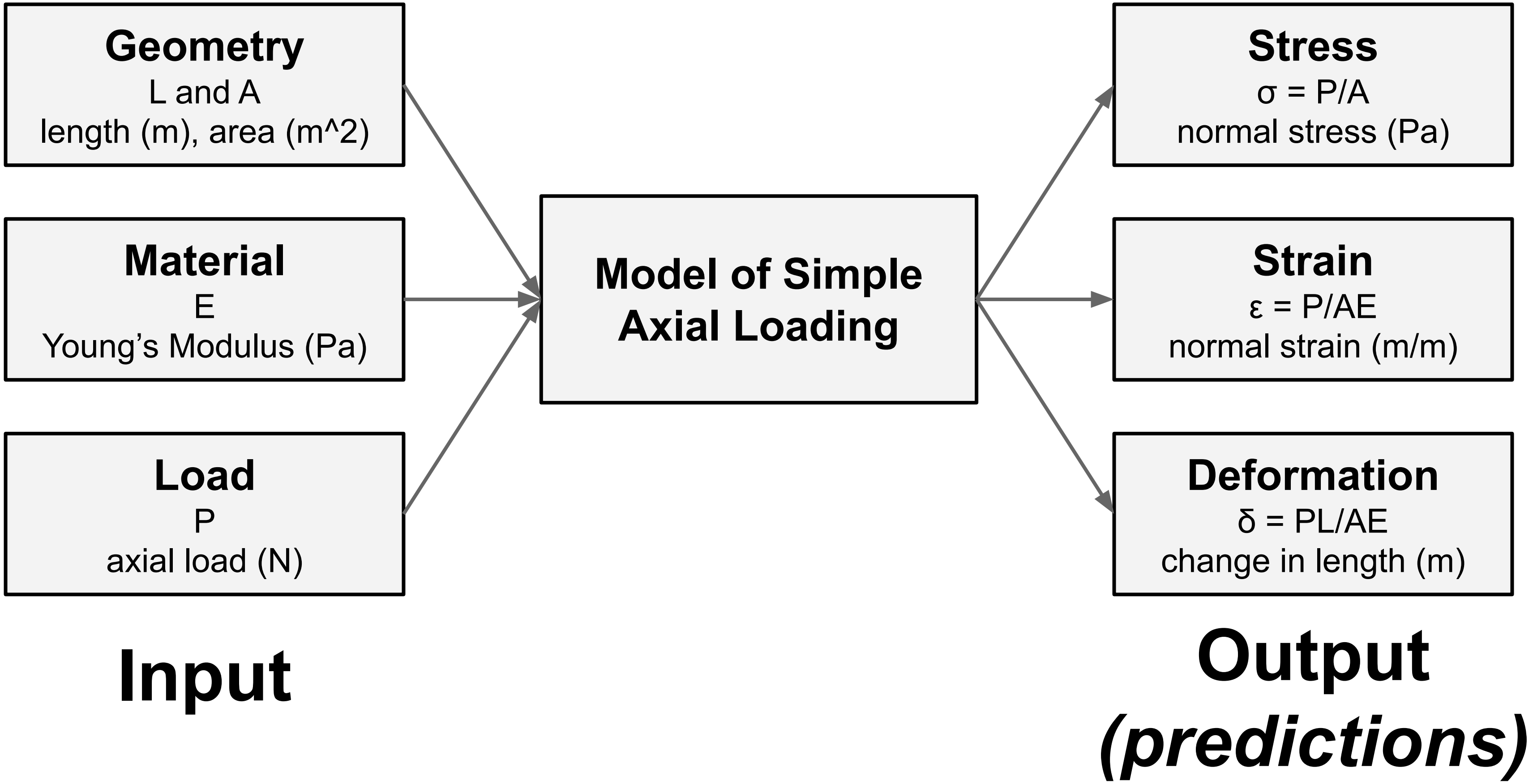
Prof. Charlie Kemp

BMED 3410: Introduction to Biomechanics
September 7, 2022
Lecture 5

Outline

1. Material sample characterization wrap-up
   a. https://sites.gatech.edu/intro-to-biomechanics/
2. Axial loading model with varying
   a. Geometry
   b. Material
   c. Load
3. An example with graphs

Model of Simple Axial Loading



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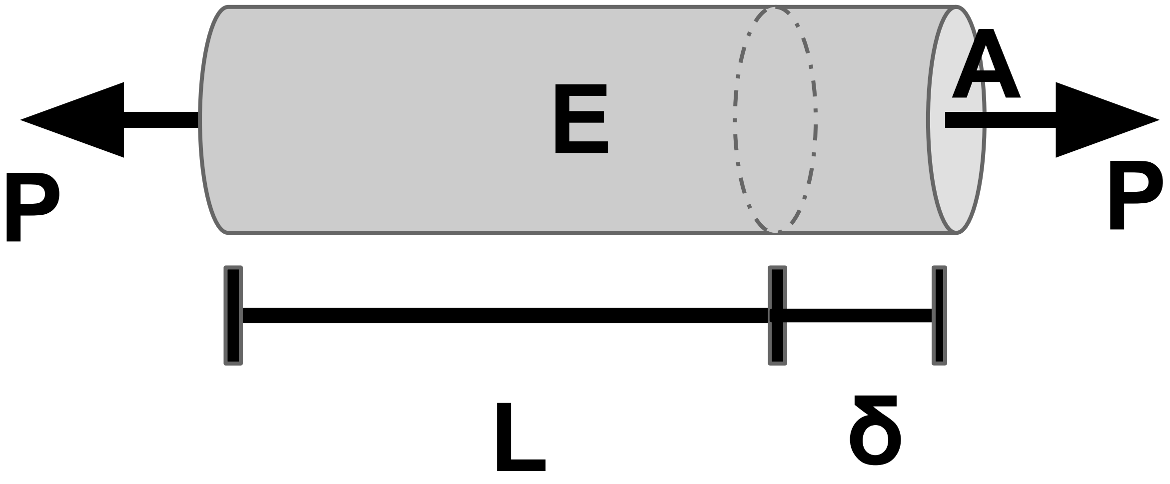
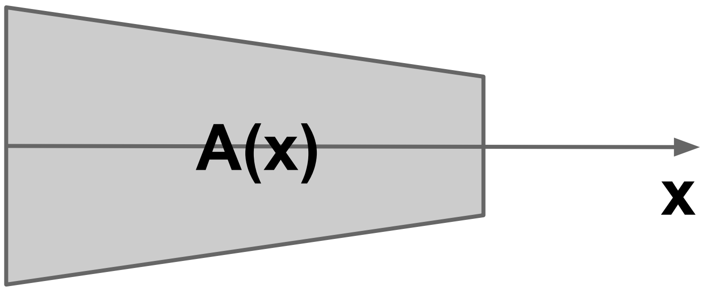
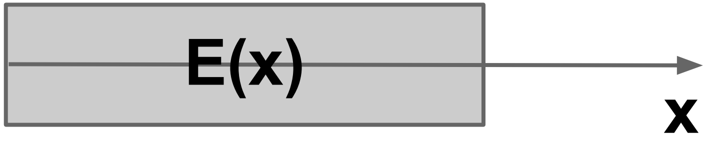


Generalize the Model for Axial Variation in

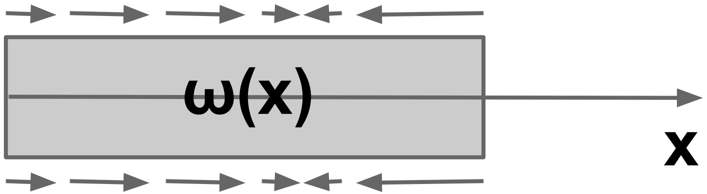
Geometry
(cross-sectional area)



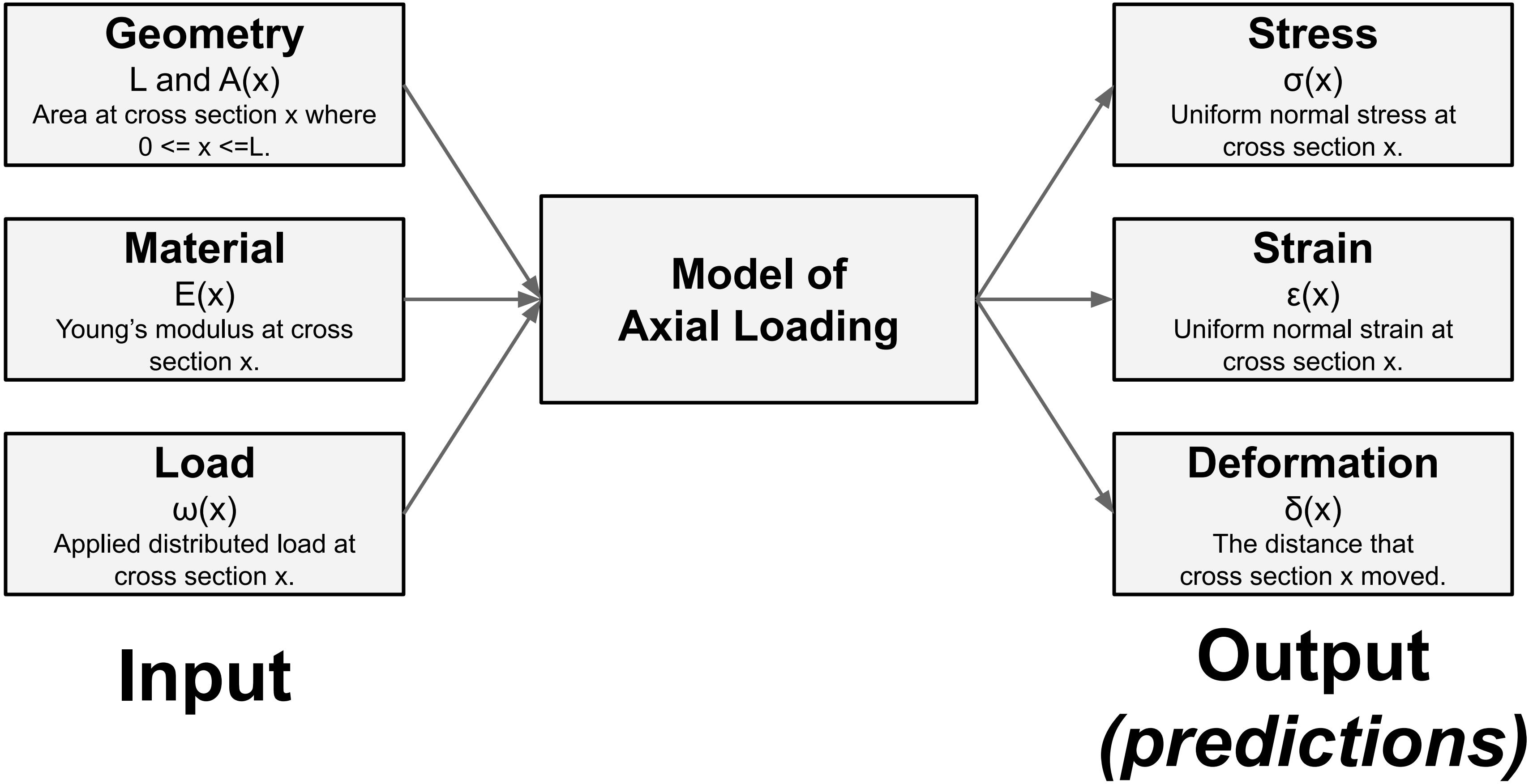
Material
(Young's modulus)



Load
(applied distributed load)



Model of Axial Loading



Observe a Real Rubber Band

- Axially varying load ($\omega(x)$)
  - Predict change in the uniformly spaced markings on a rubber band when pulled at its two ends.
  - Predict change when load applied at the ends and the middle of the rubber band.
- Axially varying cross-sectional area ($A(x)$)
  - Predict change in the uniformly spaced markings on a rubber band that has been cut to be narrow in the middle when pulled at its two ends.
  - Predict where the rubber band that has been cut to be narrow in the middle will fail.
- Axially varying material ($E(x)$)
  - Predict what happens when the rubber band is connected in series with a paperclip.

BMED 3400 Axial Loading Chart

version 1.1

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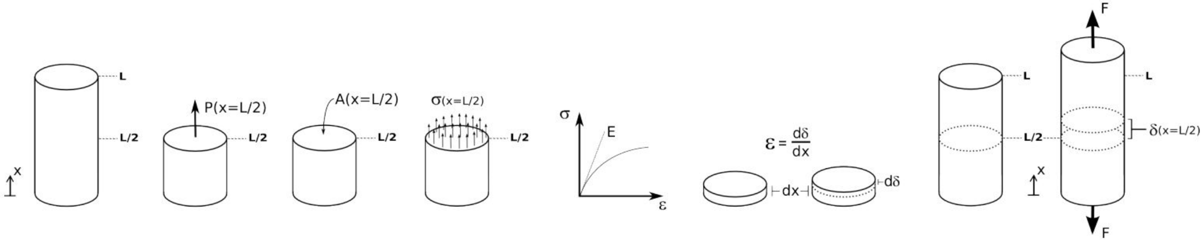
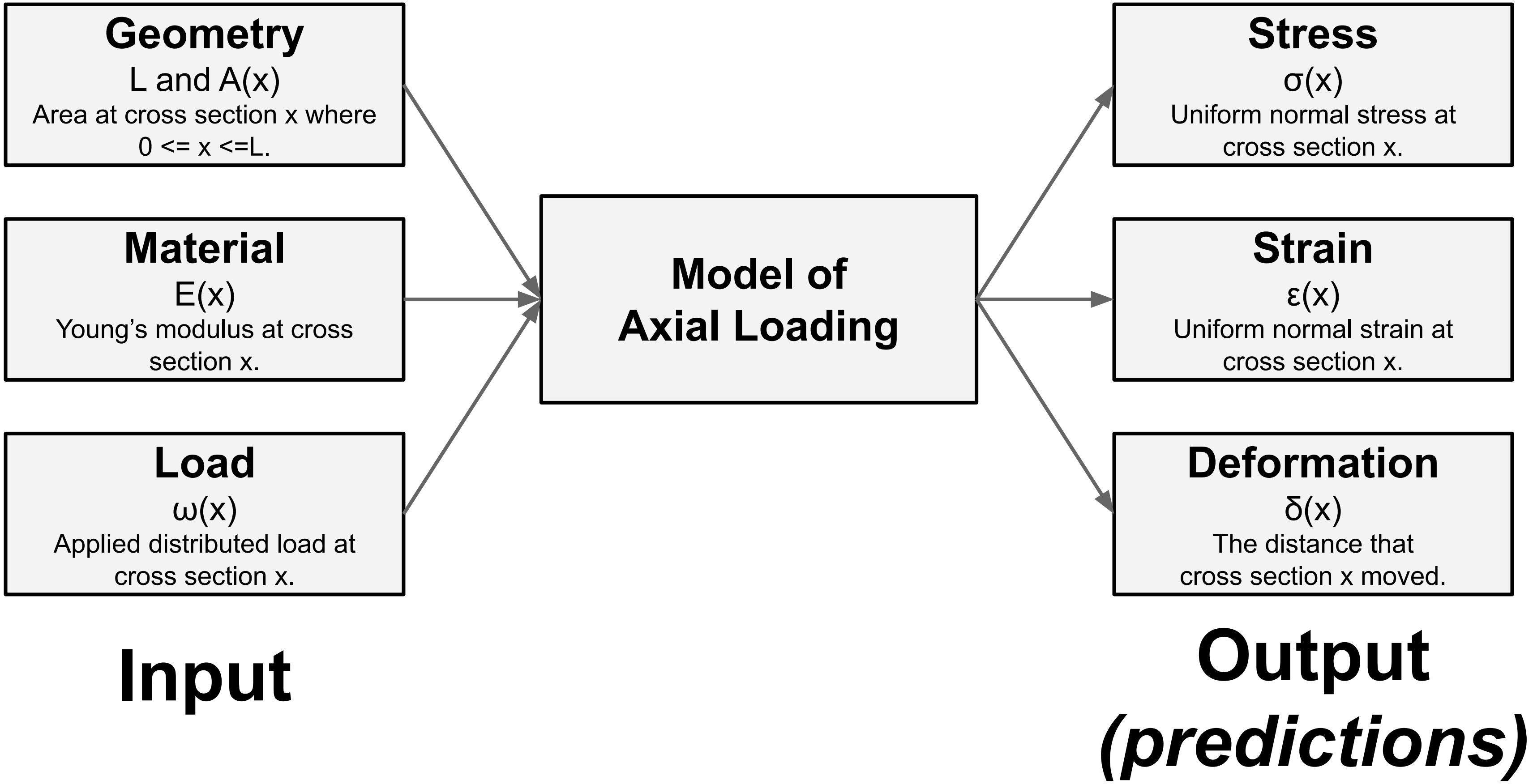
January 26, 2016

variable	*units*	*relationship to other variables*	*name*	*description*
L	meters, m		length of the object	length of axially loaded non-rigid object; model tends to work better for long narrow objects
x	meters, m	$0 \le x \le L$	location of a cross-section	location of cross section along length of object prior to loading; cross section is normal to axis of object
$\omega(x)$	$\frac{N}{m}$	$\int_{-\epsilon}^{L+\epsilon} \omega(x)\,dx = 0$ due to static equilibrium	applied distributed load	applied load points in $+x$ direction $\Rightarrow \omega(x) < 0$; applied load points in $-x$ direction $\Rightarrow \omega(x) > 0$
$P(x)$	newtons, N	$P(x) = \int_{-\epsilon}^{x} \omega(\alpha)\,d\alpha = -\int_{x}^{L+\epsilon} \omega(\alpha)\,d\alpha$	internal axial load	total internal force normal to cross section at x; $P > 0$ if cross-sectional sliver of length dx is in tension
$A(x)$	m^2	$A(x) = \int_{shape(x)} 1\,dA$	cross-sectional area	area of cross section at x
$\sigma(x)$	pascals, $\frac{N}{m^2}$	$\sigma(x) = \frac{P(x)}{A(x)}$, $\sigma = \frac{dF}{dA}$, $P(x) = \int_{A(x)} \sigma(x)\,dA$	normal stress	average normal stress across the cross section at x; uniform stress model; $\sigma > 0$ when in tension
$E(x)$	pascals, $\frac{N}{m^2}$	$E(x) = \frac{\sigma(x)}{\varepsilon(x)}$	Young's modulus	model of the material; slope of linear approximation to stress-strain curve at 0,0; stiffness of the material
$\varepsilon(x)$	unitless, %, $\frac{m}{m}$	$\varepsilon(x) = \frac{\sigma(x)}{E(x)}$, $\varepsilon(x) = \frac{P(x)}{E(x)A(x)}$, $\varepsilon = \frac{d\delta}{dx}$	normal strain	% change in length of cross-sectional sliver of length dx located at x; $\varepsilon > 0$ represents length increase
$\delta(x)$	meters, m	$\delta(x) = \int_{-\epsilon}^{x} \varepsilon(\alpha)\,d\alpha$, $\delta_{total} = \delta(L)$, $\delta(0) = 0$	deformation	distance cross section at x displaces due to load; $\delta = 0 \Rightarrow$ no change; $\delta > 0 \Rightarrow$ displaced in positive x direction



Download: https://drive.google.com/file/d/1EyWyzWyig3aHUcKi6UAXdYLLAEjpF8QO/

Model of Axial Loading



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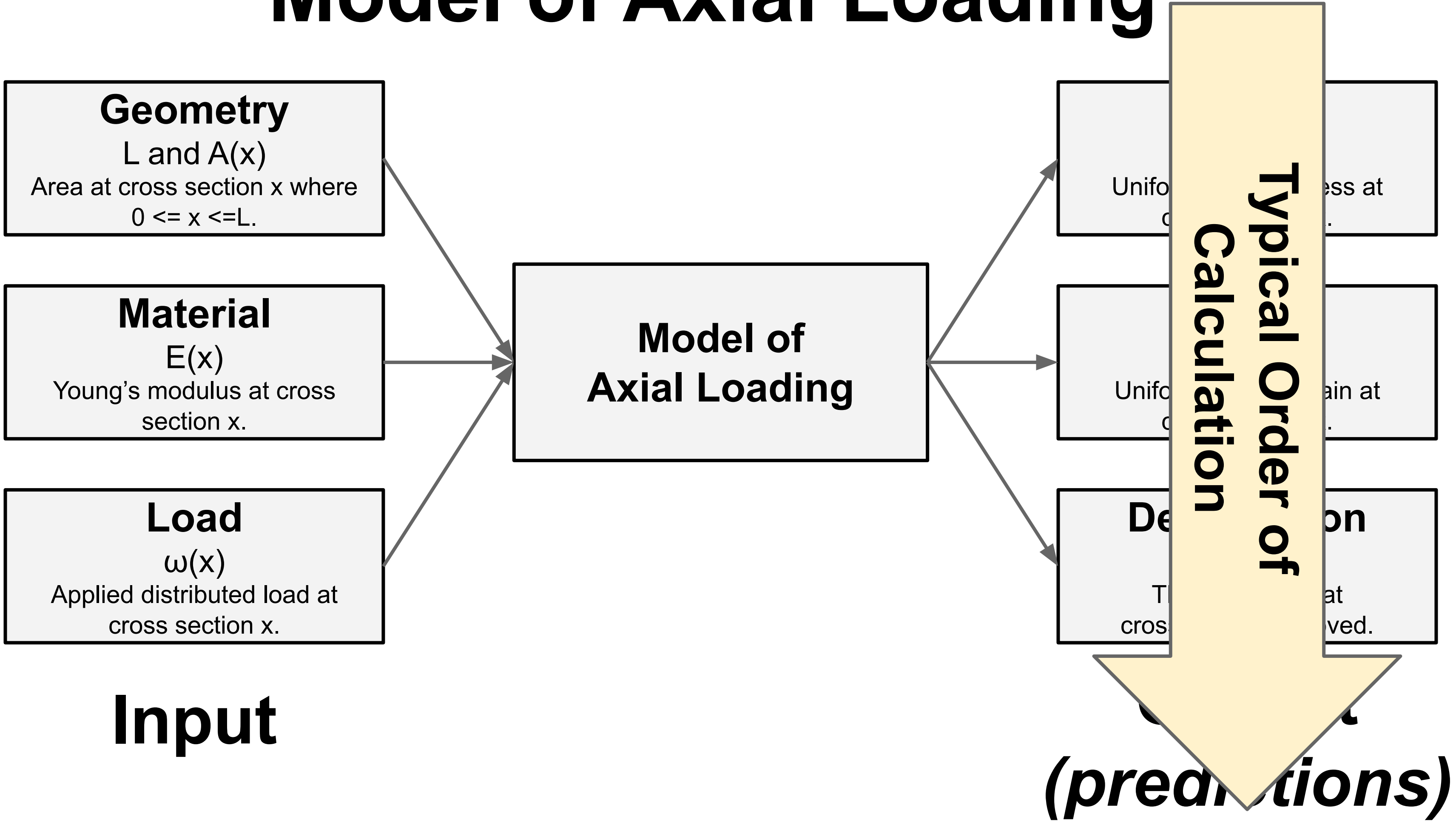

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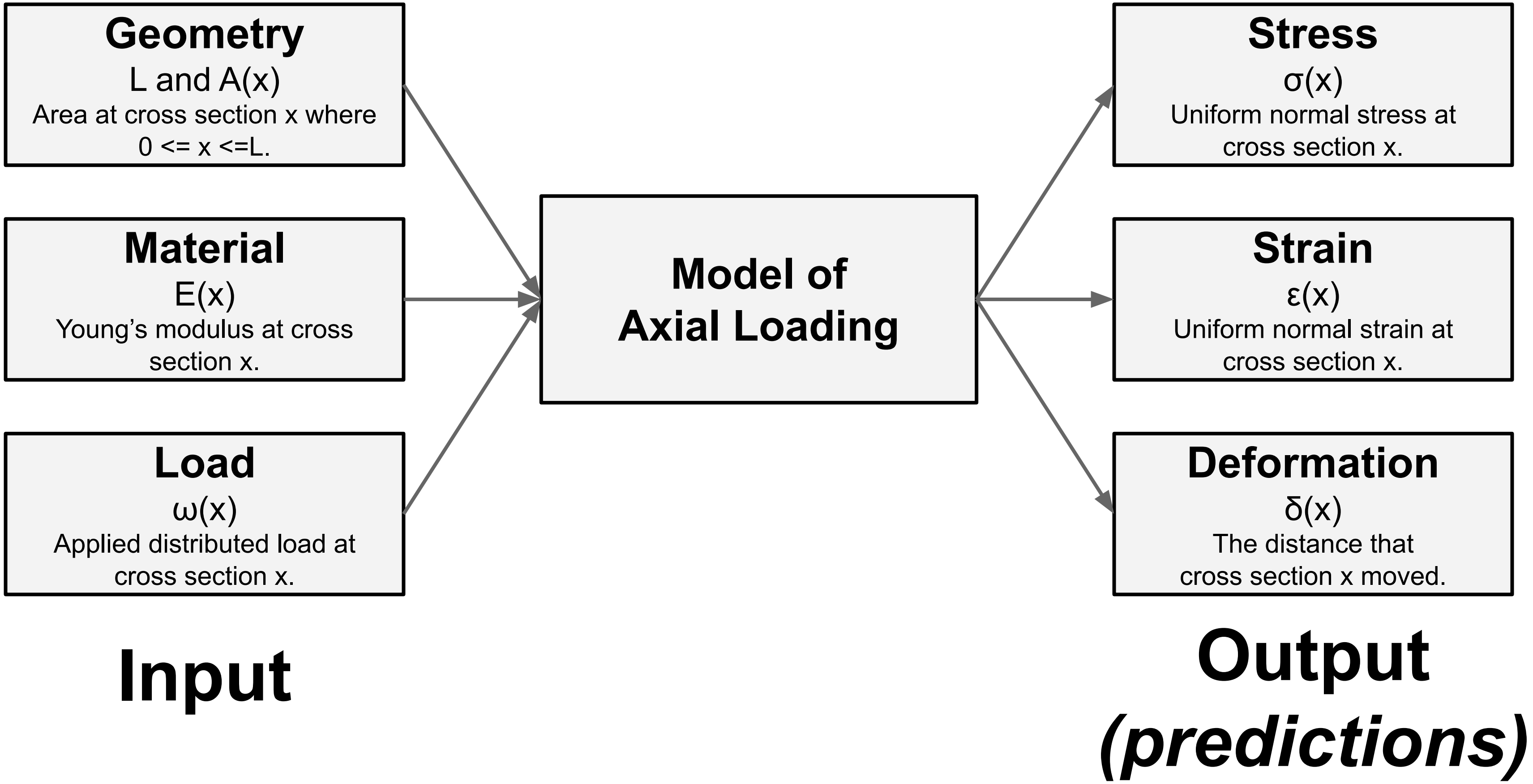

variable	*units*	*relationship to other variables*	*name*
L	meters, m		length of the object
x	meters, m	$0 \leq x \leq L$	location of a cross-section
$\omega(x)$	$\frac{N}{m}$	$\int_{-\epsilon}^{L+\epsilon} \omega(x)\, dx = 0$ due to static equilibrium	applied distributed load
$P(x)$	newtons, N	$P(x) = \int_{-\epsilon}^{x} \omega(\alpha)\, d\alpha = -\int_{x}^{L+\epsilon} \omega(\alpha)\, d\alpha$	internal axial load
$A(x)$	m^2	$A(x) = \int_{shape(x)} 1\, dA$	cross-sectional area
$\sigma(x)$	pascals, $\frac{N}{m^2}$	$\sigma(x) = \frac{P(x)}{A(x)}$, $\sigma = \frac{dF}{dA}$, $P(x) = \int_{A(x)} \sigma(x)\, dA$	normal stress
$E(x)$	pascals, $\frac{N}{m^2}$	$E(x) = \frac{\sigma(x)}{\varepsilon(x)}$	Young's modulus
$\varepsilon(x)$	unitless, %, $\frac{m}{m}$	$\varepsilon(x) = \frac{\sigma(x)}{E(x)}$, $\varepsilon(x) = \frac{P(x)}{E(x)A(x)}$, $\varepsilon = \frac{d\delta}{dx}$	normal strain
$\delta(x)$	meters, m	$\delta(x) = \int_{-\epsilon}^{x} \varepsilon(\alpha)\, d\alpha$, $\delta_{total} = \delta(L)$, $\delta(0) = 0$	deformation

Typical Order of Calculation

v	*units*	*relationship to other variables*	*name*
	meters, m		length of the object
	meters, m	$0 \leq x \leq L$	location of a cross-section
	$\frac{N}{m}$	$\int_{-\epsilon}^{L+\epsilon} \omega(x)\, dx = 0$ due to static equilibrium	applied distributed load
	newtons, N	$P(x) = \int_{-\epsilon}^{x} \omega(\alpha)\, d\alpha = -\int_{x}^{L+\epsilon} \omega(\alpha)\, d\alpha$	internal axial load
	m^2	$A(x) = \int_{shape(x)} 1\, dA$	cross-sectional area
	pascals, $\frac{N}{m^2}$	$\sigma(x) = \frac{P(x)}{A(x)}$, $\sigma = \frac{dF}{dA}$, $P(x) = \int_{A(x)} \sigma(x)\, dA$	normal stress
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	unitless, %, $\frac{m}{m}$	$\varepsilon(x) = \frac{\sigma(x)}{E(x)}$, $\varepsilon(x) = \frac{P(x)}{E(x)A(x)}$, $\varepsilon = \frac{d\delta}{dx}$	normal strain
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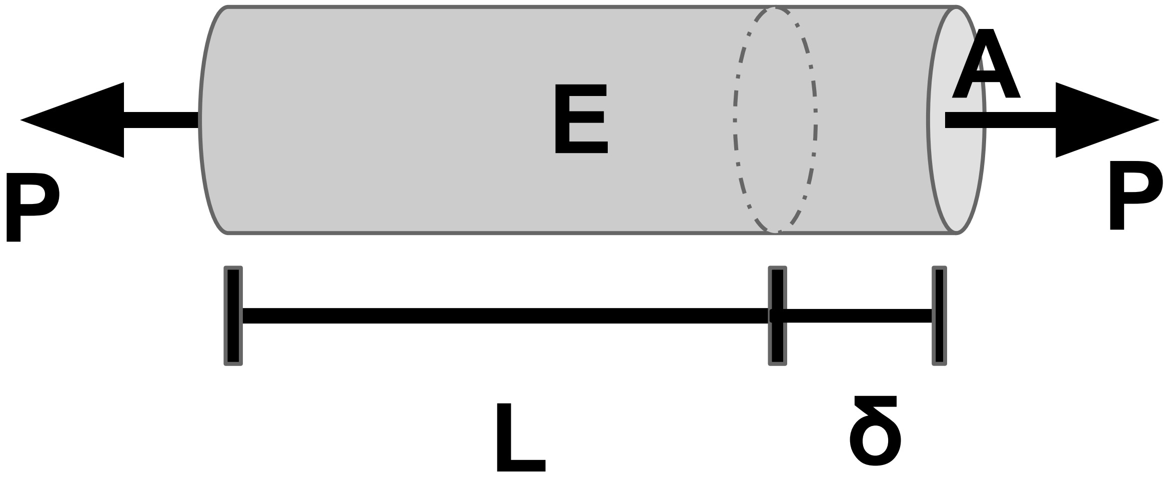
$\delta(x)$	meters, m	$\delta(x) = \int_{-\epsilon}^{x} \varepsilon(\alpha)\, d\alpha$, $\delta_{total} = \delta(L)$, $\delta(0) = 0$	deformation

Use Piecewise Low-order Polynomial Functions

- Use piecewise polynomial functions to quickly create models and graph key quantities
- http://en.wikipedia.org/wiki/Piecewise
- Generate the functions by integrating
  - dirac delta
  - constant
  - linear
  - quadratic
  - …
- Draw graphs instead of integrating with symbols

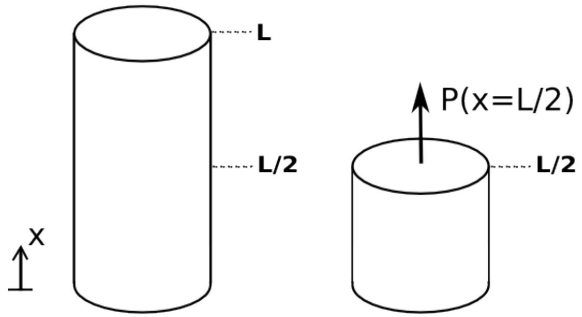
Example #1

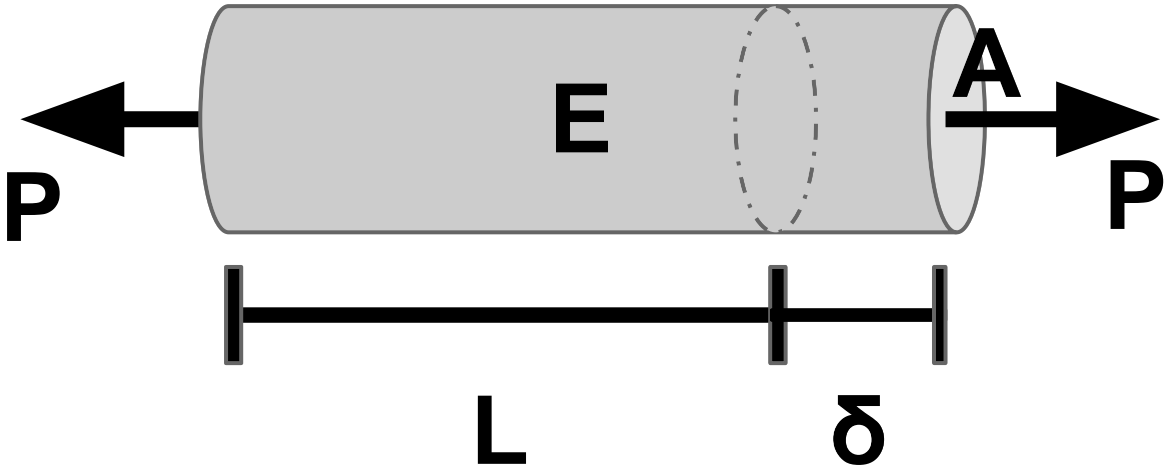


From Applied Distributed Load to the Internal Axial Load

- ω(x) represents the applied distributed load
- integrate it to find the internal axial load, P(x)

$\omega(x)$	$\frac{N}{m}$	$\int_{-\epsilon}^{L+\epsilon} \omega(x)\, dx = 0$ due to static equilibrium
$P(x)$	newtons, N	$P(x) = \int_{-\epsilon}^{x} \omega(\alpha)\, d\alpha = -\int_{x}^{L+\epsilon} \omega(\alpha)\, d\alpha$




P
E
A
P
L
δ

How can we represent a concentrated load with a distribution?

- Dirac delta function
  - "The Dirac delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,"

Not deformation! →

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

  - "and which is also constrained to satisfy the identity"

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Quotes and equations from Dirac delta function Wikipedia page.

More About the Dirac Delta Function

“integration of the delta function results in the Heaviside step function”

$$H(x) := \int_{-\infty}^{x} \delta(s)\, ds$$

“The distributional derivative of the Heaviside step function is the Dirac delta function”

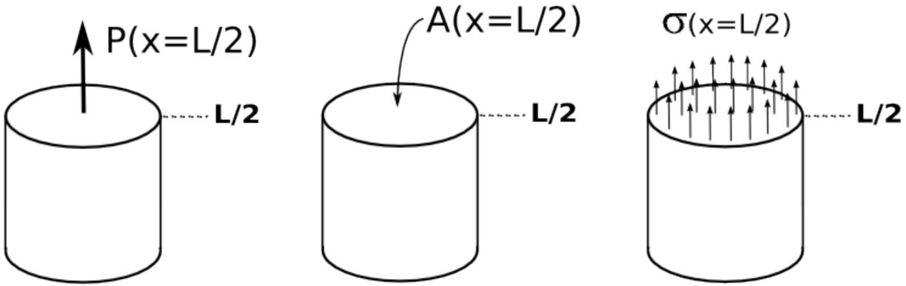
$$\frac{dH(x)}{dx} = \delta(x)$$

Quotes and equations from Heaviside step function Wikipedia page.

From Internal Axial Load to Stress

- To find stress, σ(x), simply divide the internal axial load, P(x), by the area, A(x).

$P(x)$	newtons, N	$P(x) = \int_{-\epsilon}^{x} \omega(\alpha)\, d\alpha = -\int_{x}^{L+\epsilon} \omega(\alpha)\, d\alpha$
$A(x)$	m^2	$A(x) = \int_{shape(x)} 1\, dA$
$\sigma(x)$	pascals, $\frac{N}{m^2}$	$\sigma(x) = \frac{P(x)}{A(x)}$, $\sigma = \frac{dF}{dA}$, $P(x) = \int_{A(x)} \sigma(x)\, dA$



From Stress to Strain

- To find strain, ε(x), simply divide stress, σ(x), by Young’s modulus, E(x).

$\sigma(x)$	pascals, $\frac{N}{m^2}$	$\sigma(x) = \frac{P(x)}{A(x)}$, $\sigma = \frac{dF}{dA}$, $P(x) = \int_{A(x)} \sigma(x)\, dA$
$E(x)$	pascals, $\frac{N}{m^2}$	$E(x) = \frac{\sigma(x)}{\varepsilon(x)}$
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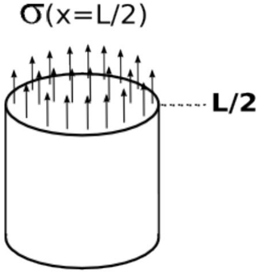
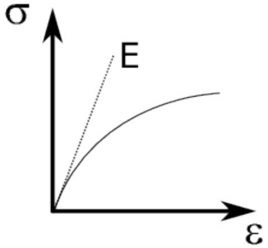
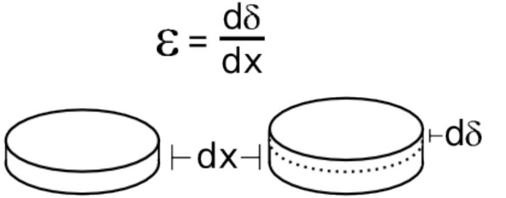


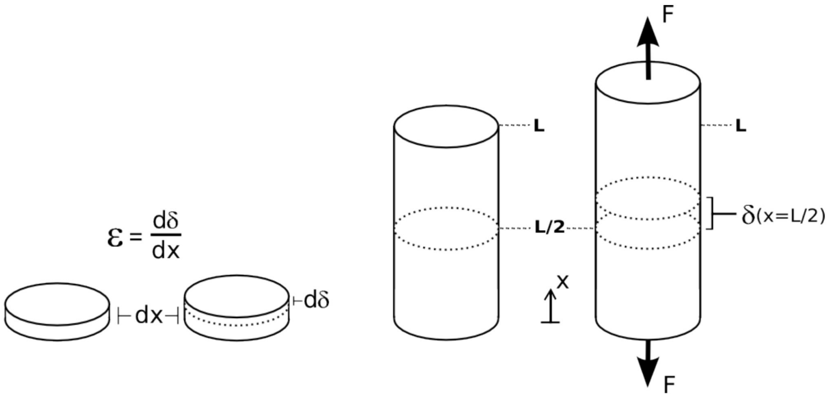




From Strain to Deformation

- Integrate the strain, ε(x), to find how far the cross section at x has moved, δ(x).

$\varepsilon(x)$	unitless, %, $\frac{m}{m}$	$\varepsilon(x) = \frac{\sigma(x)}{E(x)}$, $\varepsilon(x) = \frac{P(x)}{E(x)A(x)}$, $\varepsilon = \frac{d\delta}{dx}$
$\delta(x)$	meters, m	$\delta(x) = \int_{-\epsilon}^{x} \varepsilon(\alpha)\,d\alpha$, $\delta_{total} = \delta(L)$, $\delta(0) = 0$



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