Review for Exam #1

Prof. Charlie Kemp

BMED 3410: Introduction to Biomechanics
September 23, 2022
Lecture 10

Details about Exam #1

- In class on Wednesday, September 28
- Bring your BuzzCard
- 24% of your grade
- Full class period minus time to distribute and collect the exam
- Only a primitive writing utensil (e.g., no calculator)
- You will be given a copy of the equation sheet
- Covers all material up to and including PSS 5 next week

Materials to Study Include

- Homework #1 & Homework #2 (**on Canvas**)
- The homework quizzes
- The PSS problems
- The lectures and lecture notes
- The equation sheet and axial loading chart
- Old problems with solutions
  - **Don’t look at the static indeterminacy problems, since they’ll be covered in PSS 5**

Know Your Equation Sheet

BMED 3400 Equation Sheet
version 2.0 (May 3, 2016)

Rigid Body Model in Static Equilibrium

$\sum \vec{F} = 0$
$\sum \vec{M} = 0$

Models of Materials

$\varepsilon = \frac{L'-L}{L} = \frac{\Delta x}{x} \qquad \varepsilon > 0 \Rightarrow tension$
$\nu = -\frac{\varepsilon_{transverse}}{\varepsilon_{axial}}$
$\sigma = \frac{dF}{dA}$
$\sigma = E\varepsilon$
$\tau = \frac{dF}{dA}$
$\tau = G\gamma$

Simple Model of Axial Loading

$\sigma = \frac{P}{A} \qquad P > 0 \Rightarrow tension \qquad P < 0 \Rightarrow compression$
$\varepsilon = \frac{P}{AE}$
$\delta = \frac{PL}{AE}$

Model of Axial Loading

$\int_{\epsilon}^{L+\epsilon} \omega(x)\,dx = 0$
$P(x) = \int_{\epsilon}^{x} \omega(\beta)\,d\beta$
$\sigma(x) = \frac{P(x)}{A(x)}$
$\varepsilon(x) = \frac{\sigma(x)}{E(x)}$
$\delta(x) = \int_{\epsilon}^{x} \varepsilon(\beta)\,d\beta, \quad \delta_{total} = \delta(x = L), \quad \delta(x = 0) = 0$
$\varepsilon_{therm}(x) = \alpha(x)\triangle T(x)$
$\delta_{therm}(x) = \int_{-\epsilon}^{x} \varepsilon_{therm}(\beta)\,d\beta$

Simple Model of Twisting (Torsion)

$\tau = \frac{T}{J}r$
$\gamma = \frac{T}{GJ}r$
$\phi = \frac{TL}{GJ}$

Model of Twisting (Torsion)

$\int_{-\epsilon}^{L+\epsilon} \omega(x)\,dx = 0$
$T(x) = \int_{-\epsilon}^{x} \omega(\beta)\,d\beta = -\int_{x}^{L+\epsilon} \omega(\beta)\,d\beta$
$T(x) = \int_{shape(x)} r\tau(x,r)\,dA$
$\tau(x,r) = \frac{T(x)}{J(x)}r$
$\gamma(x,r) = \frac{\tau(x,r)}{G(x)}$
$\phi(x) = \int_{-\epsilon}^{x} \frac{1}{r}\gamma(\beta,r)\,d\beta \qquad r \neq 0$

Model of Bending

$V(x) = \int_{-\epsilon}^{x} \omega_{shear}(\beta)\,d\beta$
$M(x) = \int_{-\epsilon}^{x} \omega_{moment}(\beta) + V(\beta)\,d\beta$
$M(x) = \int_A dM = \int_A y\,dF = \int_A y\sigma(x,y)\,dA$
$\sigma(x,y) = \frac{-M(x)y}{I}$
$\varepsilon(x,y) = \frac{-M(x)y}{E(x)I}$
$\varepsilon(x,y) = \frac{-y}{\rho(x)}$
$\rho(x) = \frac{E(x)I}{M(x)}$
$\kappa(x) = \frac{1}{\rho(x)} = \frac{\frac{d^2v}{dx^2}}{\left(1+\left(\frac{dv}{dx}\right)^2\right)^{\frac{3}{2}}} \approx \frac{d^2v}{dx^2} \quad if \ \frac{dv}{dx} \ll 1$
$\frac{d^2v}{dx^2} = \ddot{\nu}(x) = \frac{M(x)}{E(x)I}$
$\nu(x) = \int\int \frac{M(x)}{E(x)I}\,dx$

Geometric Quantities for Deformable Models

$X_{CoM} = \frac{\int x\,\rho(x)\,dx}{\int \rho(x)\,dx}$
$N_{axis} = \frac{\int_A y\,dA}{\int_A dA}$
$A = \int_A dA$
$A_{circle} = \pi R^2$
$J = \int_A r^2\,dA$
$J_{circle} = \frac{\pi}{2}R^4$
$J_{ring} = J_{R_{out}} - J_{R_{in}} = \frac{\pi}{2}(R_{out}^4 - R_{in}^4)$
$I = \int_A y^2\,dA$
$I_{/axis} = I_{/centroid} + Ad^2$
$I_{rect} = \frac{bh^3}{12}$
$I_{circle} = \frac{\pi}{4}R^4$

Point Mass Model with a Circular Trajectory

$\frac{d\theta}{dt}(t) = \dot{\theta}(t) = \omega(t)$
$\frac{d^2\theta}{dt^2}(t) = \ddot{\theta}(t) = \alpha(t)$
$\vec{x}(t) = R(\cos(\theta(t))\hat{i} + \sin(\theta(t))\hat{j})$
$\vec{v}(t) = v(t)\hat{e}_{tan}(t) = R\dot{\theta}(t)\hat{e}_{tan}(t) = R\omega(t)\hat{e}_{tan}(t)$
$\|\vec{v}(t)\| = |v(t)| = \nu(t) = R|\dot{\theta}(t)| = R|\omega(t)|$
$\angle\vec{v}(t) = \begin{cases} \theta(t) + \frac{\pi}{2} & if \ \dot{\theta}(t) > 0 \\ \theta(t) - \frac{\pi}{2} & if \ \dot{\theta}(t) < 0 \end{cases}$
$\vec{a}(t) = \vec{a}_{tan}(t) + \vec{a}_{norm}(t)$

$\vec{a}(t) = \dot{v}(t)\hat{e}_{tan}(t) + \frac{v(t)^2}{R}\hat{e}_{norm}(t)$
$\vec{a}(t) = R\ddot{\theta}(t)\hat{e}_{tan}(t) + R\dot{\theta}(t)^2\hat{e}_{norm}(t)$
$\|\vec{a}_{tan}(t)\| = |\dot{v}(t)| = R|\ddot{\theta}(t)| = R|\alpha(t)|$
$\|\vec{a}_{norm}(t)\| = \frac{v(t)^2}{R} = \frac{\nu(t)^2}{R} = R\dot{\theta}(t)^2 = R\omega(t)^2$
$\vec{F}(t) = m\vec{a}(t)$

Kinematics of a Rigid Body Model

$\vec{\omega}(t) = \omega(t)\hat{k} \qquad \vec{\alpha}(t) = \alpha(t)\hat{k}$
$\vec{x}_{A/C}(t) = \vec{x}_{B/C}(t) + \vec{x}_{A/B}(t) = arbitrary + circular$
$\vec{x}_{A/C}(t) = \vec{x}_{B/C}(t) + R(\cos(\theta(t)+\beta)\hat{i} + \sin(\theta(t)+\beta)\hat{j})$
$\vec{v}_{A/C}(t) = \vec{v}_{B/C}(t) + R\omega(t)\hat{e}_{tan}(t)$
$\vec{v}_{A/C}(t) = \vec{v}_{B/C}(t) + (\vec{\omega}(t) \times \vec{x}_{A/B}(t))$
$\vec{a}_{A/C}(t) = \vec{a}_{B/C}(t) + R\alpha(t)\hat{e}_{tan}(t) + R\omega(t)^2\hat{e}_{norm}(t)$
$\vec{a}_{A/C}(t) = \vec{a}_{B/C}(t) + (\vec{\alpha}(t) \times \vec{x}_{A/B}(t)) + (\vec{\omega}(t) \times (\vec{\omega}(t) \times \vec{x}_{A/B}(t)))$

Simple Dynamics of a Rigid Body Model

$\vec{F}_{cm}, \vec{M}_{/cm}$: resultant force & moment at center of mass (cm)
$\vec{a}_{cm/C}$: center of mass accel. with respect to inertial frame (C)
$\vec{F}_{cm} = m\vec{a}_{cm/C}$
$\vec{M}_{/cm} = I_{/cm}\vec{\alpha}$

Dynamics of a Rigid Body Model

$\vec{F}_C, \vec{M}_{/C}$: resultant force & moment at inertial frame (C)

pure rotational acceleration around C:
$\vec{M}_{/C} = I_{/C}\vec{\alpha}$

rotational and translational acceleration with respect to C:

$$\begin{aligned} \vec{a}_{cm/C} &= \vec{a}_{B/C} + \vec{a}_{cm/B} \\ &= \vec{a}_{B/C} + \left(\vec{a}_{cm/B}\right)_{tangent} + \left(\vec{a}_{cm/B}\right)_{normal} \\ &= \sum \vec{a}_i \end{aligned}$$

$\pm M_{/C} = \pm I_{/cm}\alpha + \sum \pm ma_i d_i$

$$\begin{aligned} \vec{M}_{/C} &= I_{/cm}\vec{\alpha} + \vec{x}_{cm/C} \times m\vec{a}_{cm/C} \\ &= I_{/cm}\vec{\alpha} + \vec{x}_{cm/C} \times m\sum \vec{a}_i \end{aligned}$$

Rotational Inertia for Rigid Body Dynamics

$I = \int r^2 dm = \int r^2 \rho dV$
$I = I_{/cm} + md^2$

sphere	$I_{/cm} = \frac{2}{5}mR^2$
flat rectangle (⊥ to rectangle)	$I_{/cm} = \frac{1}{12}m\left(W^2 + H^2\right)$
cylinder (‖ to central axis)	$I_{/cm} = \frac{1}{2}mR^2$
cylinder (⊥ to central axis)	$I_{/cm} = \frac{1}{12}m\left(3R^2 + L^2\right)$
slender rod (⊥ to central axis)	$I_{/cm} = \frac{1}{12}mL^2$

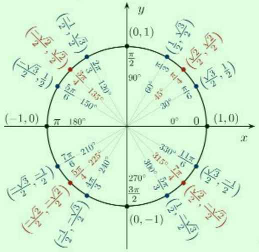
Dot Product and Cross Product Properties

$\frac{d(\vec{p}\times\vec{q})}{dt} = \vec{p}\times\dot{\vec{q}} + \dot{\vec{p}}\times\vec{q}$
$\frac{d(\vec{p}\cdot\vec{q})}{dt} = \vec{p}\cdot\dot{\vec{q}} + \dot{\vec{p}}\cdot\vec{q}$
$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{i} = \hat{j}$
$\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$
$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

$$\begin{aligned} \vec{p}\times\vec{q} &= \det\left(\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{bmatrix}\right) \\ &= (p_y q_z - p_z q_y)\,\hat{i} + (p_z q_x - p_x q_z)\,\hat{j} + (p_x q_y - p_y q_x)\,\hat{k} \\ &= |\vec{p}|\,|\vec{q}|\sin(\theta)\,\hat{n} \end{aligned}$$

$\vec{p}\cdot\vec{q} = |\vec{p}|\,|\vec{q}|\cos(\theta) = p_x q_x + p_y q_y + p_z q_z$

Unit Circle for Trigonometric Calculations & SI Prefixes



exa	10^{18}	E
peta	10^{15}	P
tera	10^{12}	T
giga	10^{9}	G
mega	10^{6}	M
kilo	10^{3}	k
hecto	10^{2}	h
deca	10^{1}	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p
femto	10^{-15}	f
atto	10^{-18}	a

[1] downloaded from https://en.wikipedia.org/wiki/Unit_circle on 4/12/16

Download the Equation Sheet: https://drive.google.com/drive/folders/1R3T1dEBK7wu6soVPAIs4BMOqPa8ML-Yl

Advice Based on Homework Quiz #1

- Use scientific notation
- Check your units
- Isolate the component of interest
- Know the difference
  - linear elastic deformation (Young’s modulus)
  - elastic deformation
  - plastic deformation (yield strength)
  - material failure (ultimate tensile strength)

Examples of Basic Skills and Concepts

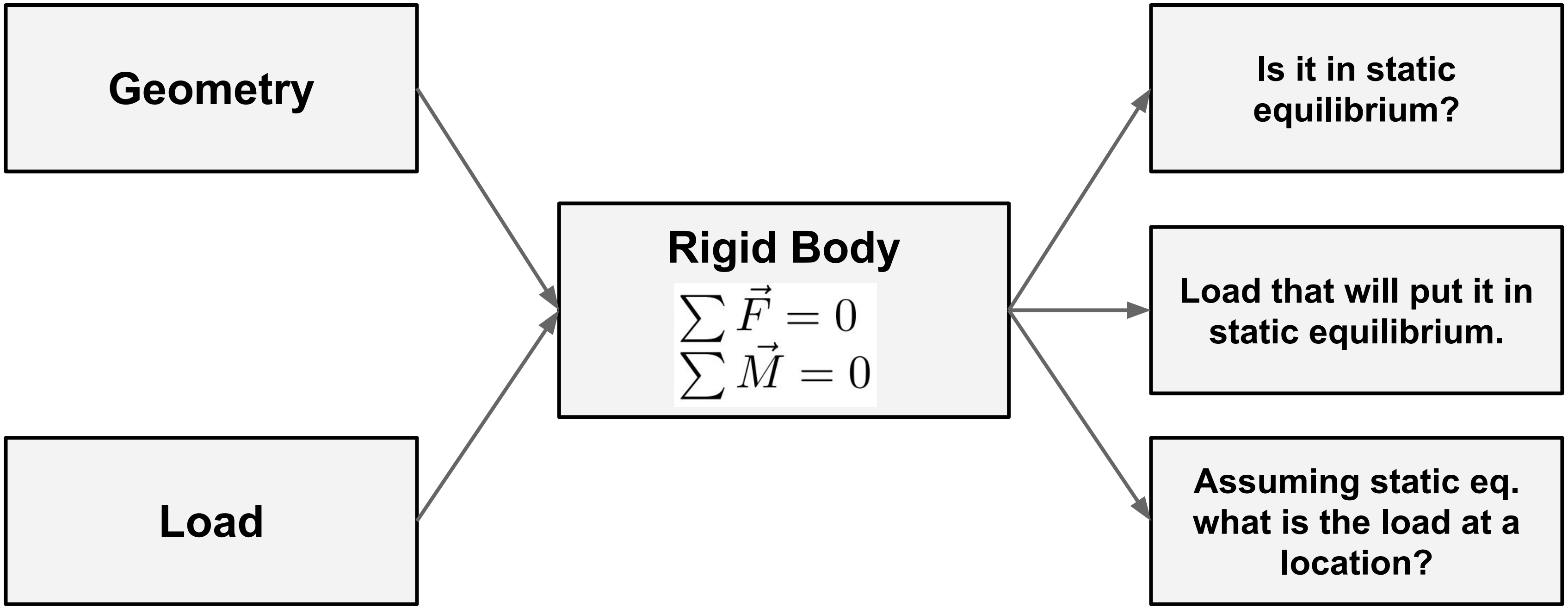
- Rigid bodies and static equilibrium
  - put in static equilibrium
  - test if body is in static equilibrium
- Free-body diagrams (FBDs)
- Stress and strain
- Stress-strain curve
  - elastic deformation, plastic deformation, failure, stress, strain, linear, non-linear, …
  - common values for Young's modulus
- Axial loading models
  - structure of our models (input, output, units, parameter names, …)
  - qualitative interpretation of how quantities change
  - graphing

Goal: Learn to Solve Real Biomedical Engineering Problems with **Simple Mechanical Models**

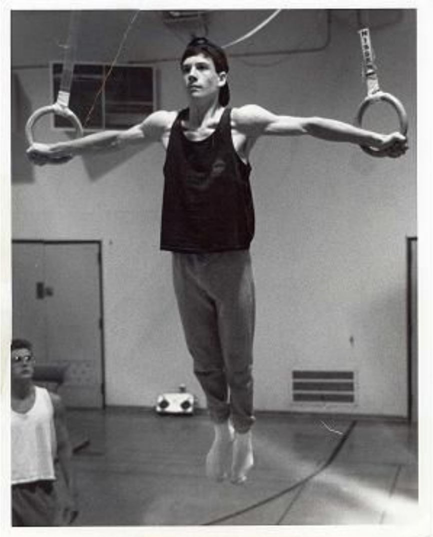
Simple Models So Far

- Rigid Body Model in Static Equilibrium
- Simple Axial Loading Model
- Axial Loading Model
- Axial Loading Model with Static Indeterminacy

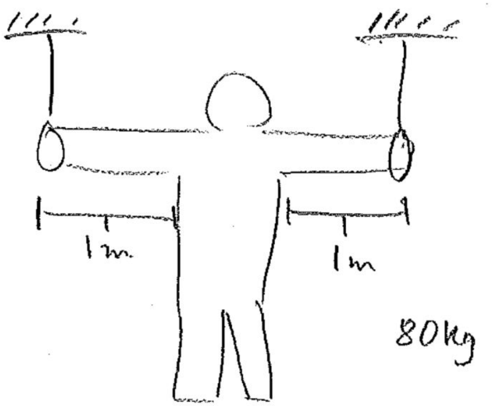
Rigid Body Model in Static Equilibrium



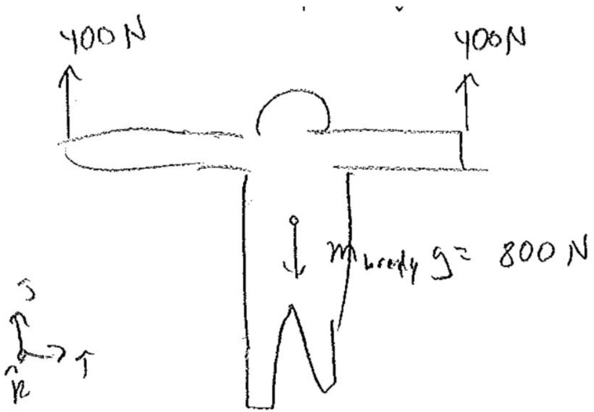
Rigid Body Model in Static Equilibrium



https://commons.wikimedia.org/wiki/File:Example2ofironcross.jpg

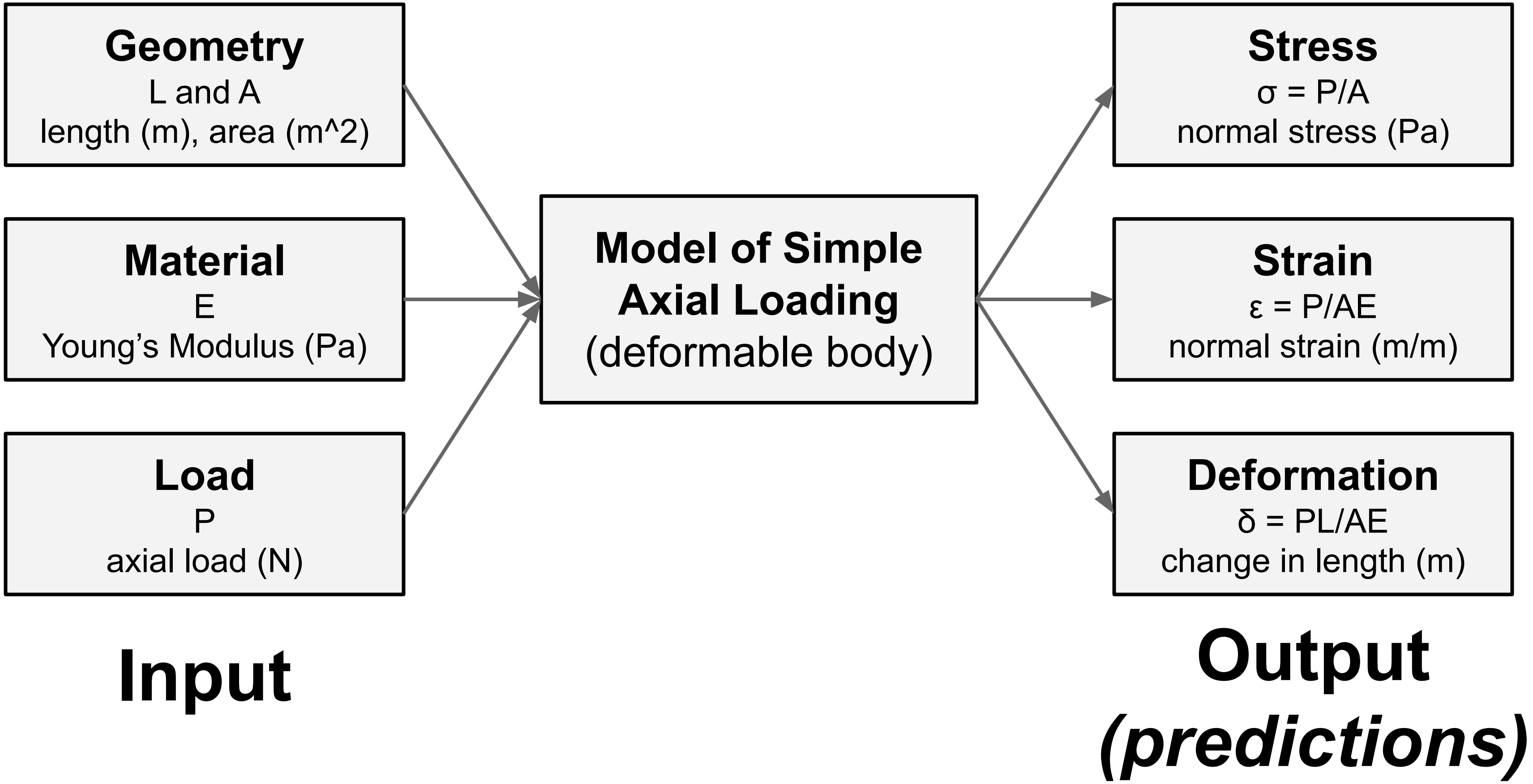


Overall Diagram

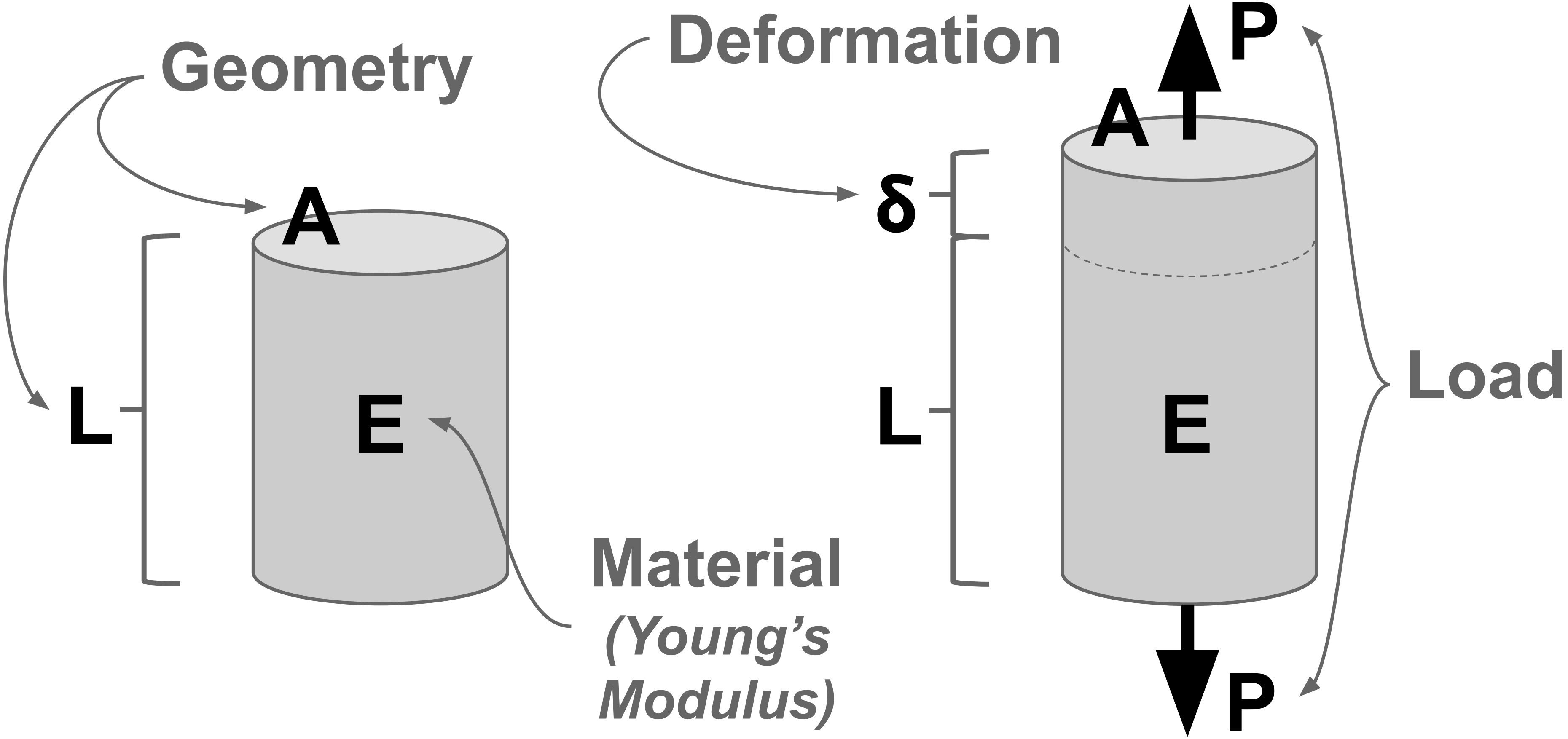


Free Body Diagram (FBD)

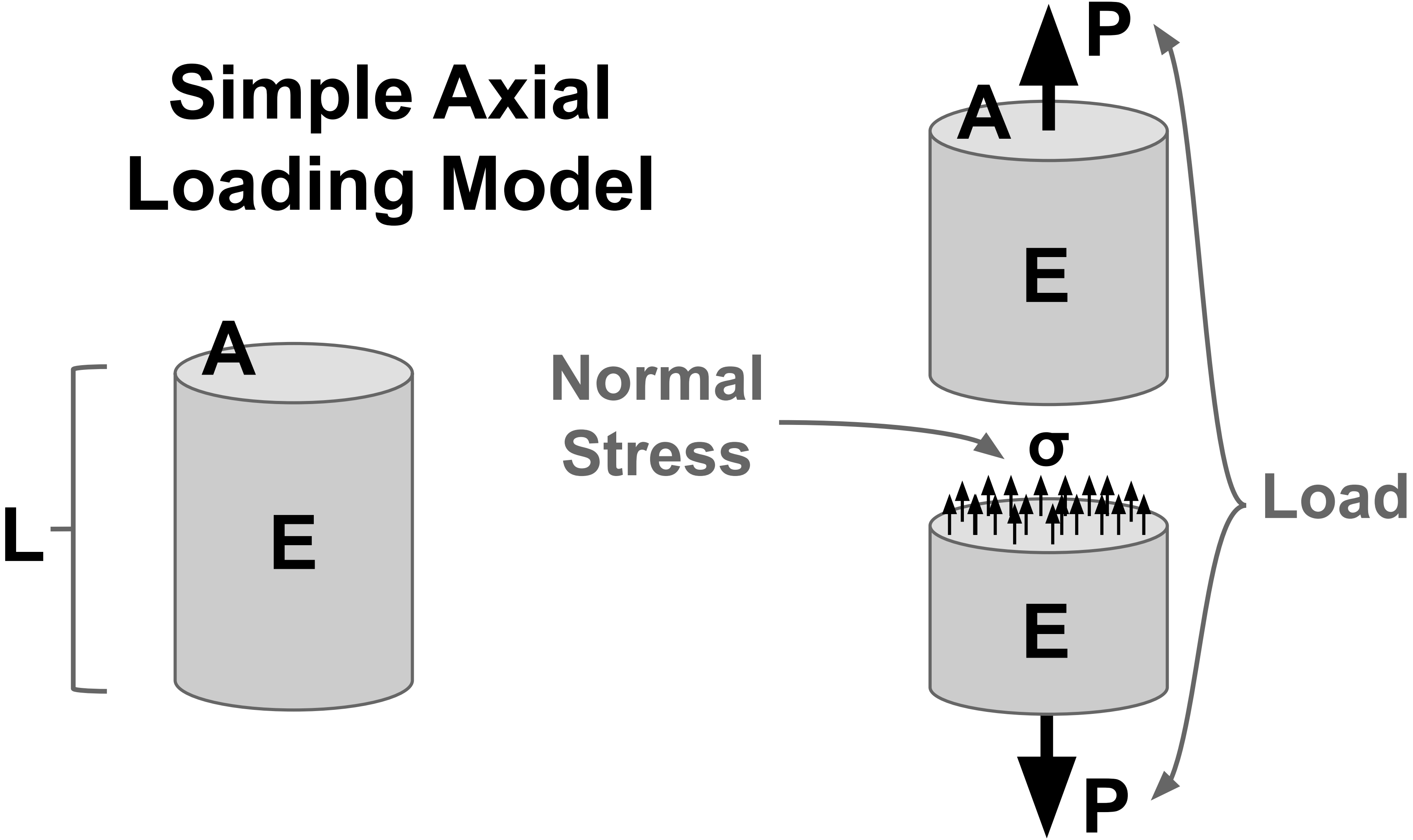
Simple Axial Loading Model



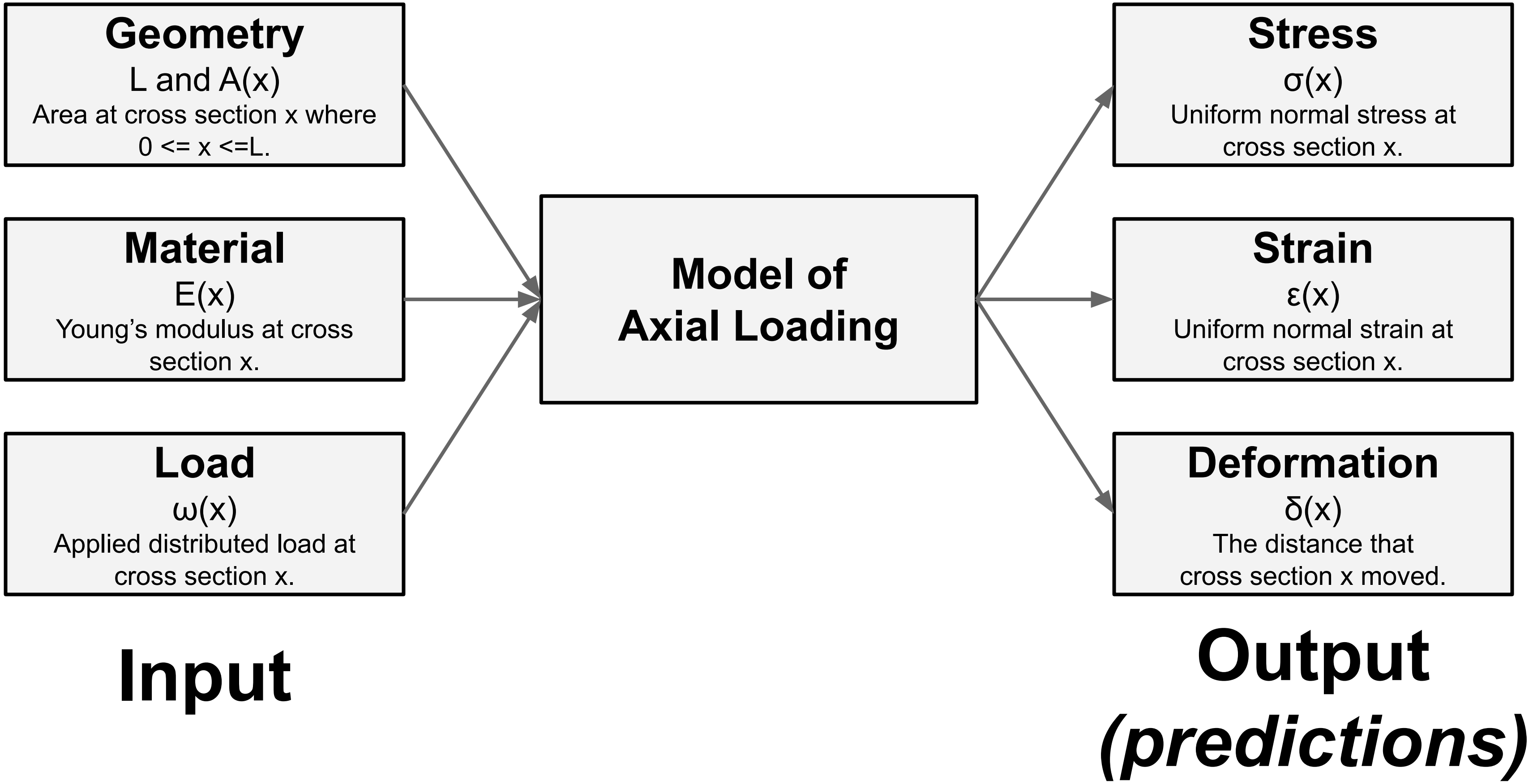
Simple Axial Loading Model



Simple Axial Loading Model



Axial Loading Model



Axial Loading Model

variable	*units*	*relationship to other variables*	*name*
L	meters, m		length of the object
x	meters, m	$0 \leq x \leq L$	location of a cross-section
$\omega(x)$	$\frac{N}{m}$	$\int_{-\epsilon}^{L+\epsilon} \omega(x)\, dx = 0$ due to static equilibrium	applied distributed load
$P(x)$	newtons, N	$P(x) = \int_{-\epsilon}^{x} \omega(\alpha)\, d\alpha = -\int_{x}^{L+\epsilon} \omega(\alpha)\, d\alpha$	internal axial load
$A(x)$	m^2	$A(x) = \int_{shape(x)} 1\, dA$	cross-sectional area
$\sigma(x)$	pascals, $\frac{N}{m^2}$	$\sigma(x) = \frac{P(x)}{A(x)},\ \sigma = \frac{dF}{dA},\ P(x) = \int_{A(x)} \sigma(x)\, dA$	normal stress
$E(x)$	pascals, $\frac{N}{m^2}$	$E(x) = \frac{\sigma(x)}{\varepsilon(x)}$	Young's modulus
$\varepsilon(x)$	unitless, %, $\frac{m}{m}$	$\varepsilon(x) = \frac{\sigma(x)}{E(x)},\ \varepsilon(x) = \frac{P(x)}{E(x)A(x)},\ \varepsilon = \frac{d\delta}{dx}$	normal strain
$\delta(x)$	meters, m	$\delta(x) = \int_{-\epsilon}^{x} \varepsilon(\alpha)\, d\alpha,\ \delta_{total} = \delta(L),\ \delta(0) = 0$	deformation

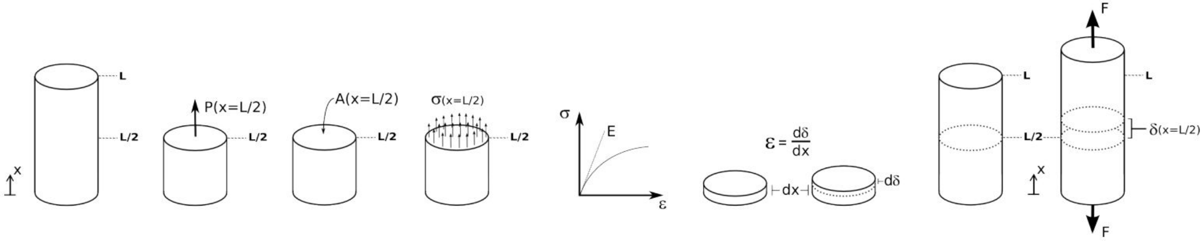
BMED 3400 Axial Loading Chart

version 1.1

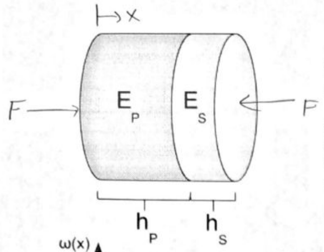
Prof. Charles C. Kemp, PhD

January 26, 2016

variable	*units*	*relationship to other variables*	*name*	*description*
L	meters, m		length of the object	length of axially loaded non-rigid object; model tends to work better for long narrow objects
x	meters, m	$0 \leq x \leq L$	location of a cross-section	location of cross section along length of object prior to loading; cross section is normal to axis of object
$\omega(x)$	$\frac{N}{m}$	$\int_{-\epsilon}^{L+\epsilon} \omega(x)\,dx = 0$ due to static equilibrium	applied distributed load	applied load points in $+x$ direction $\Rightarrow \omega(x) < 0$; applied load points in $-x$ direction $\Rightarrow \omega(x) > 0$
$P(x)$	newtons, N	$P(x) = \int_{-\epsilon}^{x} \omega(\alpha)\,d\alpha = -\int_{x}^{L+\epsilon} \omega(\alpha)\,d\alpha$	internal axial load	total internal force normal to cross section at x; $P > 0$ if cross-sectional sliver of length dx is in tension
$A(x)$	m^2	$A(x) = \int_{shape(x)} 1\,dA$	cross-sectional area	area of cross section at x
$\sigma(x)$	pascals, $\frac{N}{m^2}$	$\sigma(x) = \frac{P(x)}{A(x)}$, $\sigma = \frac{dF}{dA}$, $P(x) = \int_{A(x)} \sigma(x)\,dA$	normal stress	average normal stress across the cross section at x; uniform stress model; $\sigma > 0$ when in tension
$E(x)$	pascals, $\frac{N}{m^2}$	$E(x) = \frac{\sigma(x)}{\varepsilon(x)}$	Young's modulus	model of the material; slope of linear approximation to stress-strain curve at 0,0; stiffness of the material
$\varepsilon(x)$	unitless, %, $\frac{m}{m}$	$\varepsilon(x) = \frac{\sigma(x)}{E(x)}$, $\varepsilon(x) = \frac{P(x)}{E(x)A(x)}$, $\varepsilon = \frac{d\delta}{dx}$	normal strain	% change in length of cross-sectional sliver of length dx located at x; $\varepsilon > 0$ represents length increase
$\delta(x)$	meters, m	$\delta(x) = \int_{-\epsilon}^{x} \varepsilon(\alpha)\,d\alpha$, $\delta_{total} = \delta(L)$, $\delta(0) = 0$	deformation	distance cross section at x displaces due to load; $\delta = 0 \Rightarrow$ no change; $\delta > 0 \Rightarrow$ displaced in positive x direction



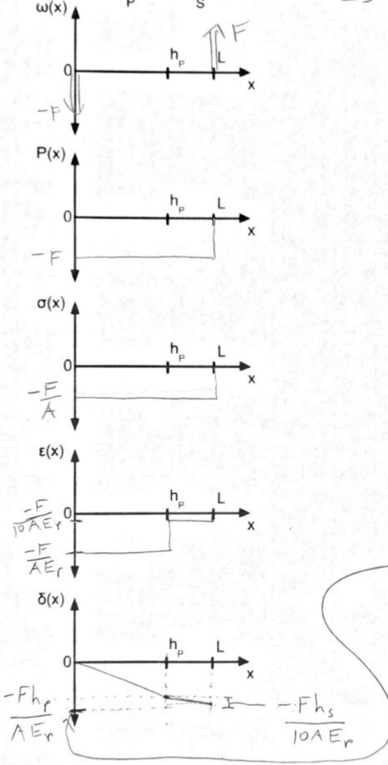
Download: https://drive.google.com/file/d/1EyWyzWyig3aHUcKi6UAXdYLLAEjpF8QO/



1) Complete the free body diagram (FBD) to the left by drawing the loads on the tissue column when the finger pushes on the surface with force F.

2) Draw graphs for the five functions shown with axes on the left of the page. Label key values for full credit. **Only use variables from the table** when labeling your graphs. ***Do not use values from the table.***

$E_s = 10 E_p$



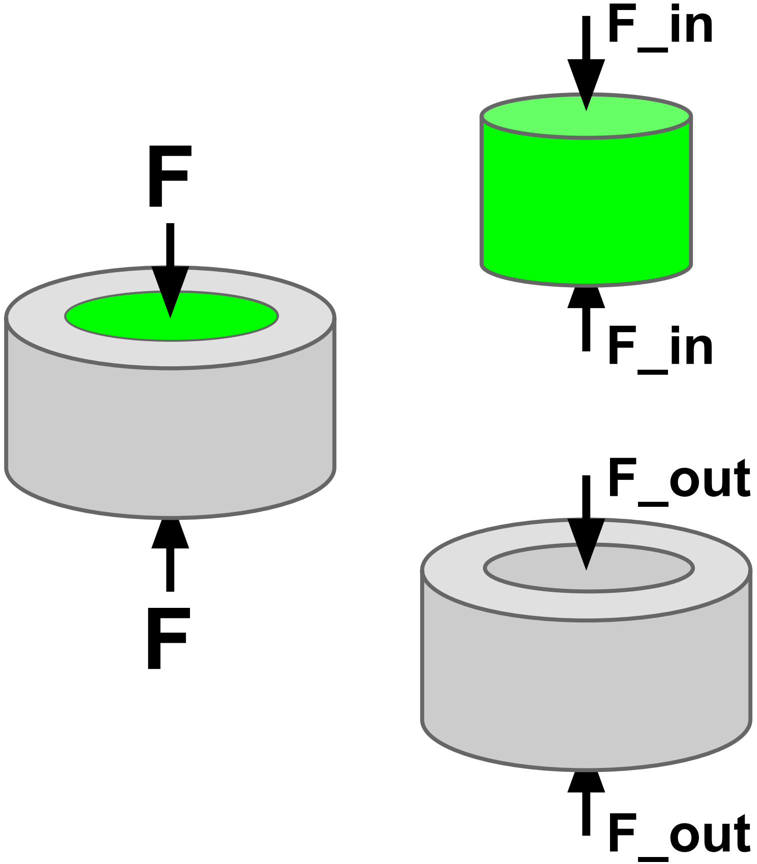
$h_p = 2h_s$

$$\delta(x=L) = \delta_{total} = \frac{-Fh_p}{AE_p} + \frac{-Fh_s}{10AE_p}$$

$$= \frac{-F}{AE_p}\left(h_p + \frac{h_p}{20}\right)$$

$$\delta_{total} = \frac{-Fh_p}{AE_p}\left(1 + \frac{1}{20}\right)$$

Axial Loading Model with Static Indeterminacy



Statics is Not Enough

$F = F_{in} + F_{out}$

Solve with Deformations

$\delta_{in} = \delta_{out}$

Avoiding “careless errors” & “stupid mistakes”

1. Achieve proficiency with basic skills, so you have more time to read carefully, think, and check your answers. For example, you should rarely need the equation sheet.
2. While studying, make sure not to look at the solutions until you are fully done working on a problem.
3. Work through the homework, homework quizzes, and old problems.
4. Use the full time allotted to take the exam.
5. Write clearly.
6. Don’t skip steps.
7. Check your units.
8. Work through problems in more than one way.
9. Use visualization, diagrams, and graphs to check that the symbols on the page make sense.
10. Try to go backwards through your math. This is similar to proofreading a paper backwards.
11. At every step, ask yourself what the various values and variables mean in terms of the original problem. Does it match what you would expect? If not, why?

My Last Lecture

- Thanks for making it enjoyable
- Prof. Scott Hollister starts a week from today

Photo from https://bme.gatech.edu/bme/faculty/Scott-Hollister