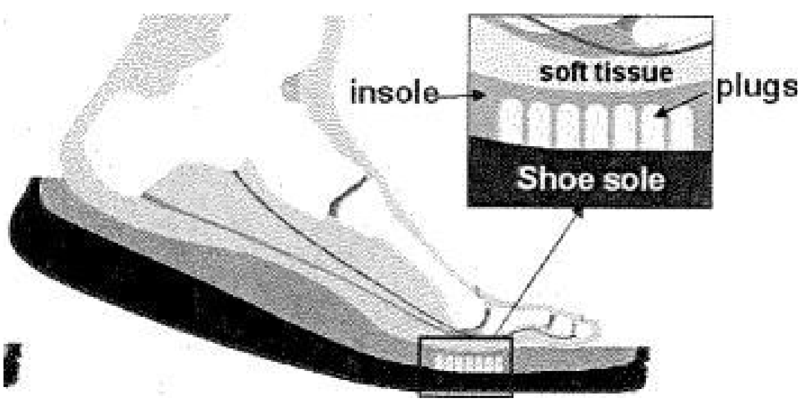
Problem #3 (33 points)

Over time, pressure across a person's foot can lead to pressure ulcers, which can result in complications such as infection and damage to muscle and bone [1]. People with diabetes and peripheral neuropathy have higher risk of acquiring pressure ulcers on their feet. One approach to mitigating this risk is to use a specialized insole placed within a shoe that sits between the bottom of the foot and the the interior of the shoe (see figure below from [2]). Researchers have provided evidence that patient-specific insoles that are stiffer in some regions and more compliant in others can beneficially alter the pressure distribution across a person's foot [2].

You are designing a system that uses a 3D printer to create custom patient-specific insoles that are more compliant in some areas than others. However, your printer can only print with two materials that can be well-modeled as linear elastic with Young's moduli E_s and E_c. To change the effective stiffness of a region of the insole, you propose to print rectangular boxes using the more compliant material, E_c, embedded within the stiffer material, E_s. This will result in insoles similar to the hand-made insoles with compliant plugs from [2] shown in the figure below.

You have chosen to model a region of the 3D-printed insole as being independent from all other regions. For a region with cross-sectional area A_r, your system prints compliant rectangular boxes with a total cross-sectional area $A_c = \alpha A_r$, where α is a positive real number between 0 and 1, inclusive. If $\alpha = 1$, the entire region of the insole is made of the compliant material. If $\alpha = 0$, the entire region is made of the stiffer material. Thus, the total cross-sectional area of the stiffer material in the region is $A_s = (1 - \alpha)A_r$. The cross-sectional areas A_r, A_c, and A_s remain constant from the bottom cross section to the top cross section of the insole region. When you apply various loads to the region, the compliant rectangular boxes and the stiffer material surrounding them deform by approximately the same amount. Without an applied load, the height of the insole is L_r, which is equal to the height of each compliant rectangular box, L_c, and the height of the stiffer material, L_s. Thus, $L_r = L_c = L_s$.

Given your design, you expect a region to behave as though it were fabricated with a distinct material with Young's modulus E_r. For this problem, you must derive an equation for E_r that only uses the known design parameters: E_c, E_s, α, A_r, and L_r.



For full credit, you must simplify your equation for E_r, show your work, state your assumptions, and clearly communicate your reasoning using diagrams, equations, and text. Correct equations will result in no credit unless you also clearly communicate how you arrived at them.

[1] https://en.wikipedia.org/wiki/Pressure_ulcer
[2] Actis, R. L., Ventura, L. B., Lott, D. J., Smith, K. E., Commean, P. K., Hastings, M. K., & Mueller, M. J., Multi-plug insole design to reduce peak plantar pressure on the diabetic foot during walking. Medical & Biological Engineering & Computing, 46(4), 363–371, 2008. (http://www.ncbi.nlm.nih.gov/pmc/articles/PMC2650823/)