NRE8900 Special Problem Final Paper: Viscosity Formulisms for Two-Fluid Equations and Flux Surface Geometry Models

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1 Introduction

There have been efforts to describe the magnetized plasmas in tokamaks for almost 50 years. In doing so, two mainly different formulisms to describe viscosities in the two-fluid equations of magnetized plasmas appeared. The short mean free path(i.e., collisional or Pfisch-Schluter) description of magnetized plasma as originally formulated by Braginskii1, and Robinson and Bernstein assumes an ordering in which the ion mean flow is on the order of the ion thermal speed, $|\mathbf{V}_{\perp}| << |\mathbf{V}_{\parallel}| \sim V_{th_i}$. By short-mean free path, we're considering the collisional effects in tokamak plasmas. Mikhailovskii and Tsypin realized that this ordering is not the one of most interest in many practical situations in which the flow is weaker and on the order of the ion diamagnetic heat flux divided by the pressure. In their drift ordering the ion flow velocity is assumed to be on the order of the diamagnetic drift velocity, $|\mathbf{V}_{\perp}| \sim |\mathbf{V}_{\parallel}| \sim V_{th_i}$, the case of interest for most magnetic confinement and fusion devices in general, and the edge of many tokamaks in particular.[1] As a consequence, the basic difference in two different approaches appears in whether or not retaining the heat transfer terms in the viscosity formulism.

Independent from the viscosity formulism, another important aspect in describing the tokamak plasma is in the application of flux surface geometry models. For the earlier works, the difficulty in representing the complex shape of flux surfaces was greatly simplified with the elliptical model to decrease the amount of analytical and computational efforts. The actual flux surface shape, however, is far different from the simple elliptical model. Therefore it is also of great interest to solve the two-fluid equations with more realistic flux surface model. A decade ago, Miller et al. developed a flux surface model with more detailed desciption of the actual D-shaped tokamak plasmas, which we call "The Miller Equilibrium Model(or Miller Model)" in this paper.

This paper discusses the two different approaches in deriving the viscosity formulism and presents the final forms of each viscosity formulism and their relations to each other. Also presented are two flux surface models, the Elliptical Model and the Miller Model.

2 Two-fluid Plasma Equations

[6][4]

Before we begin, it is helpful to revisit the two-fluid equations that are used in describing magnetized plasmas, shown below.

Continuity Equation:

$$\frac{\partial n_j}{\partial t} + \overrightarrow{\nabla} \cdot (n_j \overrightarrow{V}_j) = S_j^0 \tag{1}$$

Momentum Balance Equation:

$$m_j \frac{\partial}{\partial t}(n_j \overrightarrow{V}_j) + \overrightarrow{\nabla} \cdot \overleftrightarrow{M} = n_j e_j (\overrightarrow{E} + \overrightarrow{V}_j \times \overrightarrow{B}) + \overrightarrow{R}_j^1 + \overrightarrow{S}_j^1 \tag{2}$$

and Energy Balance Equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2} Tr M_j\right) + \nabla \cdot \overrightarrow{Q}_j = n_j e_j \overrightarrow{V}_j \cdot \overrightarrow{E} + R_j^2 + S_j^2 = W_j + \overrightarrow{V}_j \cdot (\overrightarrow{R}_j^1 + n_j e_j \overrightarrow{E}) + S_j^2 \tag{3}$$

where,

$$\overleftrightarrow{M} = n_j m_j \overrightarrow{V}_j \overrightarrow{V}_j + \overleftrightarrow{P}_j = n_j m_j \overrightarrow{V}_j \overrightarrow{V}_j + \frac{1}{3}(Tr P_j)\overleftrightarrow{I} + \overleftrightarrow{\prod}_j \tag{4}$$

$$\overleftrightarrow{P}_j = \frac{1}{3}(Tr P_j)\overleftrightarrow{I} + \overleftrightarrow{\prod}_j \tag{5}$$

$$\frac{1}{2} Tr M_j = \frac{3}{2} P_j + \frac{1}{2} n_j m_j \overrightarrow{V}_j^2 = \frac{3}{2} n_j T_j + \frac{1}{2} n_j m_j \overrightarrow{V}_j^2 \tag{6}$$

$$\overrightarrow{Q}_j = \left(\frac{5}{2} n_j T_j + \frac{1}{2} n_j m_j V_j^2\right) \overrightarrow{V}_j + \overleftrightarrow{\prod}_j \cdot \overrightarrow{V}_j + \overrightarrow{q}_j \tag{7}$$

$$R_j^2 = W_j + \overrightarrow{V}_j \cdot \overrightarrow{R}'_j. \tag{8}$$

W_j for ions and electrons are given as follows:[4]

$$W_i = \frac{3 m_e n_e \nu_e}{m_i}(T_e - T_i) \tag{9}$$

$$W_e = -W_i - \overrightarrow{R}_e \cdot (\overrightarrow{V}_e - \overrightarrow{V}_i). \tag{10}$$

The two-fluid equations above are most commonly presented form in literatures due to their simplist expressions for each terms. The viscosity terms appear in the momentum balance equation, Eq. (2), and the energy balance equation, Eq. (3). To show the viscosity terms in the two-fluid equations, we can use equations (4) and (7) to rewrite the two-fluid equations as follows:

Momentum Balance Equation:

$$m_j \frac{\partial}{\partial t}(n_j \overrightarrow{V}_j) + n_j m_j (\overrightarrow{V}_j \cdot \nabla)\overrightarrow{V}_j + \overrightarrow{\nabla} P_j + \overrightarrow{\nabla} \cdot \overleftrightarrow{\prod}_j = n_j Z_j e(\overrightarrow{E} + \overrightarrow{V}_j \times \overrightarrow{B}) - \overrightarrow{R}^1 + \overrightarrow{S}^1 \tag{11}$$

and Energy Balance Equation:

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_j + \frac{1}{2}n_j m_j \overrightarrow{V}_j^2\right) + \nabla \cdot \left[\left(\frac{5}{2}n_j T_j + \frac{1}{2}n_j m_j V_j^2\right)\overrightarrow{V}_j + \overleftrightarrow{\prod}_j \cdot \overrightarrow{V}_j + \overrightarrow{q}_j\right] = ne\overrightarrow{V}_j \cdot \overrightarrow{E} + R_j^2 + S_j^2 \tag{12}$$

The momentum balance equation above can be rewritten for each components of the typical plasma coordinates as follows:

$$m_j \frac{\partial}{\partial t}(n_j V_r) + n_j m_j [(\overrightarrow{V}_j \cdot \nabla)\overrightarrow{V}_j]_r + \frac{1}{h_r}\frac{\partial P_j}{\partial r} + [\nabla \cdot \overleftrightarrow{\prod}]_r = n_j e_j (E_r + V_\theta B_\phi - V_\phi B_\theta) - R_r + S_r^1 \tag{13}$$

$$m_j \frac{\partial}{\partial t}(n_j V_\theta) + n_j m_j [(\overrightarrow{V}_j \cdot \nabla)\overrightarrow{V}_j]_\theta + \frac{1}{h_\theta}\frac{\partial P_j}{\partial \theta} + [\nabla \cdot \overleftrightarrow{\prod}]_\theta = n_j e_j (E_\theta + V_r B_\phi) - R_\theta + S_\theta^1 \tag{14}$$

$$m_j \frac{\partial}{\partial t}(n_j V_\phi) + n_j m_j [(\overrightarrow{V}_j \cdot \nabla)\overrightarrow{V}_j]_\phi + [\nabla \cdot \overleftrightarrow{\prod}]_\phi = n_j e_j (E_\phi + V_r B_\theta) - R_\phi + S_\phi^1 \tag{15}$$

The momentum exchange between electron and ion fluids which connects the equations of two species is represented by either $\overrightarrow{F}$ or $\overrightarrow{R}$. Note that $\overrightarrow{F}$ and $\overrightarrow{R}$ are related by $\overrightarrow{F} = -\overrightarrow{R}$, if $\overrightarrow{F}$ is to be used in the two-fluid equations.

There are two different approaches in presenting the viscosities in the euqations above, one by Braginskii and another by Mikhilovskii. As a consequence of different ordering approaches by Braginskii and Mikhailovskii, only these viscosity terms have different forms in solving the two-fluid equations. We will now discuss the differences in these two different viscosity formulisms and presents the results in the following section.

3 Two Approaches to Viscosity Formulism

3.1 Braginskii vs. Mikhilovskii Formulism: History of Ordering Viewpoints

The differences in viscosity formulism can be best understood from the ordering point of view. The earliest work by Braginskii adopted an ordering in which the flow velocity is assumed to be comparable to the ion thermal speed, $|\mathbf{V}_\perp| << |\mathbf{V}_\parallel| \sim V_{th_i}$.[1] As a consequence, the high flow ordering of Braginskii ignores heat flow corrections to the pressure anisotropy, the gyroviscosity, and the collisional perpendicular viscosity.[2] Later, Mikhailovskii and Tsypin have developed their formulism for the case of non-negligible poloidal velocity, i.e., thermal velocity on the same order as of rotational and poloidal velocity, $|\mathbf{V}_\parallel| \sim |\mathbf{V}_\perp| \sim V_{th_i}$. They realized the shortcomings of the Braginskii's treatment and employed a more appropriate drift ordering in an attempt to retain the important heat flow modifications to the pressure tensor. [2]

Therefore, the short mean-free path two-fluid equations of Braginskii do not retain heat flux terms in the viscosity. This is a consequence of his ordering the plasma flow velocity comparable to the sound speed and considering only the lowest order correction (in the small gyroradius, $\delta = \rho/L_\perp$, and mean-free path, $\triangle = \lambda/L_\parallel$, expansion parameters) to the leading order Maxwellian distribution function. Here, $\rho = v_{th_i}/\Omega$ is the ion gyroradius, $\lambda = v_{th_i}/\nu$ is the Coulomb mean-free path, and $L_\perp$ and $L_\parallel$ are characteristic perpendicular and parallel length scales, respectively.[4] Also for this reason, most short mean free path treatments of turbulence in magnetized plasmas must use some version of the Mikhailovskii and Tsypin results to properly treat the temperature gradient terms in the viscous stress tensor. [1]

Earlier attempts of Mikhailovskii and Tsypin to retain heat flux terms in viscosity did not resolve the problem entirely since their Sonine polynomial expansion of the correction to the Maxwellian was truncated too soon, the non-linear contributions from the ion-ion collision operator were neglected, and electrons were not considered.[4] Attempts to upgrade Mikhailovskii's formulism was first made by Hazeltine but his work was also imcomplete due to improper truncation of the polynomials. Catto and Simakov recently revisited the problem, addressed these shortcomings in poynomial truncation, and derived complete drift-ordered short mean-free path equations for ions and electrons.[4]

Catto and Simakov showed that the truncated polynomial technique Mikhailovskii and Tsypin used together with their neglect of modifications to the pressure anisotropy due to the nonlinear nature of the ion-ion collision operator resulted in an erroneous expression for the collisional perpendicular viscosity because the ion distribution function was not retained to second order in the ion gyroradius and mean-free path expansions. Therefore, we cas assume that Catto and Simakov's recent results include the most accurate orderings to render viscosity formulism that can be generally applied to magnetized plasmas. The more general ordering may also be required to more fully treat neoclassical effects at the edge of a tokamak when the radial scale lengths are small enough that $\delta \sim \triangle$, as first noted by Rogister. Catto and Simakov's ordering, however, allows turbulent fluctuations to be as large as the unperturbed background plasma quantities. For the background variations, their ordering is consistent with, but more general than, the usual Pfirsch-Schluter ordering.[1] We, therefore, treat Catto and Simakov's orderings and their findings as the most generally developed formulism based on Mikhailovskii and Tsypin's approaches in this paper.

3.2 Evolution of Mikahilovskii's approach: Catto and Simakov's orderings

When Catto and Simakov failed to recover improved equations from the widely-used drift kinetic equation of Hazeltine, they realized that his kinetic equation is only exact through first order in δ. They noticed that, in addition to Hazeltine's ordering accuracy to δ, second order accuracy in δ^2 is also required to account for the effects of the Reynolds and gyroviscous stress tensors.[4] So they performed a systematic expansion of the ion kinetic equation in the small parameters δ and $\triangle$ to determine the ion distribution function to order $\delta^2 \sim \delta\triangle \sim \triangle^2$ in terms of the ion flow velocity and the parallel and diamagnetic heat fluxes($q_{\|}$ and $\overrightarrow{q}_{\perp}$).[1] With more accurate ion distribution function, they have corrected the drift kinetic equation to account for δ^2 effects and used the improved formalism to derive expressions for gyroviscosity and (ion) perpendicular viscosity for plasmas of arbitrary collisionality in terms of a few velocity integrals of the δ correction to the leading order distribution function, which is assumed isotropic in the velocity space. [4]

Catto and Simakov assumed that the collision frequency is small compared to the cyclotron frequency as in all drift orderings. Additionally, they allowed the perpendicular scale lengths($L_{\perp}$) to be much less than (as well as comparable to) the parallel ones($L_{\|}$) so that $\rho/L_{\perp}$ can be comparable to $\lambda/L_{\|}$, as is the case in many magnetic confinement applications. Their drift ordering also assumes the mean ion flow velocity V to be on the order of the diamagnetic drift velocity, which is on the order of the ion diamagnetic and collisional parallel heat fluxes q divided by the ion pressure $p_i = nT_i$, with n the ion density. In short, Catto and Simakov's orderings can be summarized in the following form:

$$\frac{|V|}{v_{th_i}} \sim \frac{|q|}{p_i v_{th_i}} \sim \delta \sim \triangle,$$

with $|V| \sim V_{\|} \sim |V_{\perp}|$.

By allowing for this possibility Catto and Simakov obtained a formulation that can safely be used to study turbulent transport in collisional plasmas, and permits stronger poloidal density, temperature, and electrostatic potential variation in tokamaks than the normal Pfirsch-Schluter ordering. [1] [2]

3.3 Braginskii's Viscosity Formulism

To be consistent with the history of the viscosity formulism development, presented here is the summary of Braginskii's formulism, which is well summarized in Stacey's book.[6]

The viscous stress tensor is in the form:

$$\overleftrightarrow{\prod} = \begin{pmatrix} \prod_{rr} & \prod_{r\theta} & \prod_{r\phi} \\ \prod_{\theta r} & \prod_{\theta\theta} & \prod_{\theta\phi} \\ \prod_{\phi r} & \prod_{\phi\theta} & \prod_{\phi\phi} \end{pmatrix}$$

.

Each elements of viscous stress tensor for a magnetized plasma can be decomposed into the corresponding rate-of-stress tensor elements in each coordinates(parallel, perpendicular, and gyroviscous terms) as follows:

$$\prod_{\alpha\beta} = \prod_{\alpha\beta}^{0} + \prod_{\alpha\beta}^{12} + \prod_{\alpha\beta}^{34}. \quad (17)$$

Each of the rate-of-stress tensor elements are again related to the elements of the traceless rate-of-strain tensor in the flux surface coordinates as follows:

$\prod_{\alpha\beta}^{0} = -\eta_0 W_{\alpha\beta}^{0}$
$\prod_{\alpha\beta}^{12} = -(\eta_1 W_{\alpha\beta}^{1} + \eta_2 W_{\alpha\beta}^{2})$
$\prod_{\alpha\beta}^{34} = \eta_3 W_{\alpha\beta}^{3} + \eta_4 W_{\alpha\beta}^{4}$

where the elements of the traceless rate-of-strain tensor are given as follows:

$$\mathrm{W}_{\alpha\beta}^{0} \equiv \frac{3}{2}\left(f_\alpha f_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)\left(f_\mu f_\nu - \frac{1}{3}\delta_{\mu\nu}\right) W_{\mu\nu}$$

$$W_{\alpha\beta}^{1} \equiv \left(\delta_{\alpha\mu}^{\perp}\delta_{\beta\nu}^{\perp} + \frac{1}{2}\delta_{\alpha\beta}^{\perp} f_\mu f_\nu\right) W_{\mu\nu}$$

$$W_{\alpha\beta}^{2} \equiv (\delta_{\alpha\mu}^{\perp} f_\beta f_\nu + \delta_{\beta\nu}^{\perp} f_\alpha f_\mu) W_{\mu\nu}$$

$$W_{\alpha\beta}^{3} \equiv \frac{1}{2}(\delta_{\alpha\mu}^{\perp}\epsilon_{\beta\gamma\nu} + \delta_{\beta\nu}^{\perp}\epsilon_{\alpha\gamma\mu}) f_\gamma W_{\mu\nu}$$

$$W_{\alpha\beta}^{4} \equiv (f_\alpha f_\mu \epsilon_{\beta\gamma\nu} + f_\beta f_\nu \epsilon_{\alpha\gamma\mu}) f_\gamma W_{\mu\nu}.$$

$\delta^{\perp}_{\alpha\beta}$ is defined as $\delta^{\perp}_{\alpha\beta} \equiv \delta_{\alpha\beta} - f_\alpha f_\beta$. The Einstein summation convention is employed in the above formulae. The Kronecker delta function ($\delta_{\alpha\beta}$) and the antisymmetric unit tensor ($\epsilon_{\alpha\beta\gamma}$) are defined as follows:

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} 1 & \text{if in right permutation} \\ 0 & \text{if in negative permutation or if } \alpha = \beta,\ \beta = \gamma,\ \alpha = \gamma \end{cases}$$

Stacey worked out the derivation of all these elements with assumptions that reflects the physics in tokamak plasmas and presents the results in Chapter 10 of his book.[6] He assumes the followings.

$$f_\psi = |B_\psi|/|B| \approx 0$$
$$f_p = |B_p|/|B| \ll 1$$
$$f_\phi = |B_\phi|/|B| \approx 1$$

He also neglects v_r since $v_r \ll v_\theta < v_\phi$ and axisymmetry (i.e., $\partial/\partial l_\phi = 0$). With the definition of A_0 as follows:

$$A_0 \equiv -\frac{1}{3}(W_{\psi\psi} + W_{pp}) + \frac{2}{3}W_{\phi\phi} + 2f_p W_{p\phi},$$

all the elements of the viscous stress tensors are as presented below. Please note that these results are the updated version of Table 10.1 in Stacey's book since a few minor errors are corrected and agreed with Stacey.

$$\prod\nolimits^0_{\psi\psi} = \frac{1}{2}\eta_0 A_0$$
$$\prod\nolimits^{12}_{\psi\psi} = \eta_1 \left[(RB_p)^{-1}\frac{\partial(RB_p v_p)}{\partial l_p} - f_p R\frac{\partial(v_\phi R^{-1})}{\partial l_p} \right]$$
$$\prod\nolimits^{34}_{\psi\psi} = -\eta_3 \left[h_p\frac{\partial(v_p h_p^{-1})}{\partial l_\psi} - f_p R\frac{\partial(v_\phi R^{-1})}{\partial l_\psi} \right]$$

$$\prod\nolimits^0_{\psi p} = \prod\nolimits^0_{p\psi} = 0$$
$$\prod\nolimits^{12}_{\psi p} = \prod\nolimits^{12}_{p\psi} = -\eta_1 h_p\frac{\partial(v_p h_p^{-1})}{\partial l_\psi} + (\eta_1 - \eta_2) f_p R\frac{\partial(v_\phi R^{-1})}{\partial l_\psi}$$
$$\prod\nolimits^{34}_{\psi p} = \prod\nolimits^{34}_{p\psi} = -\eta_3 (RB_p)^{-1}\frac{\partial(RB_p v_p)}{\partial l_p} - (\eta_4 - \eta_3) f_p R\frac{\partial(v_\phi R^{-1})}{\partial l_p}$$

$$\prod_{\psi\phi}^{0} = \prod_{\phi\psi}^{0} = 0$$
$$\prod_{\psi\phi}^{12} = \prod_{\phi\psi}^{12} = -\eta_2 R \frac{\partial(v_\phi R^{-1})}{\partial l_\psi}$$
$$\prod_{\psi\phi}^{34} = \prod_{\phi\psi}^{34} = -\eta_4 R \frac{\partial(v_\phi R^{-1})}{\partial l_p}$$

$$\prod_{pp}^{0} = \frac{1}{2}\eta_0 A_0$$
$$\prod_{pp}^{12} = -\eta_1 (RB_p)^{-1} \frac{\partial(RB_p v_p)}{\partial l_p} + (\eta_1 - 2\eta_2) f_p R \frac{\partial(v_\phi R^{-1})}{\partial l_p}$$
$$\prod_{pp}^{34} = \eta_3 h_p \frac{\partial(v_p h_p^{-1})}{\partial l_\psi} + (2\eta_4 - \eta_3) f_p R \frac{\partial(v_\phi R^{-1})}{\partial l_\psi}$$

$$\prod_{p\phi}^{0} = \prod_{\phi p}^{0} = -\frac{3}{2}\eta_0 f_p A_0$$
$$\prod_{p\phi}^{12} = \prod_{\psi p}^{12} = -\eta_2 R \frac{\partial(v_\phi R^{-1})}{\partial l_p}$$
$$\prod_{p\phi}^{34} = \prod_{\phi p}^{34} = \eta_4 R \frac{\partial(v_\phi R^{-1})}{\partial l_\psi}$$

$$\prod_{\phi\phi}^{0} = -\eta_0 A_0$$
$$\prod_{\phi\phi}^{12} = 2\eta_2 f_p R \frac{\partial(v_\phi R^{-1})}{\partial l_p}$$
$$\prod_{\phi\phi}^{34} = -2\eta_4 f_p R \frac{\partial(v_\phi R^{-1})}{\partial l_\psi}$$

3.4 Catto's Viscosity Formulism

Recalling the standard ion expressions for the parallel heat flux ($q_\|$) and diamagnetic heat flux ($\overrightarrow{q}_\perp$),

$q_\| = -(125p/32M\nu)\hat{n} \cdot \nabla T$ and

$\overrightarrow{q}_\perp = (5p/2M\Omega)\hat{n} \times \nabla T$

where $\nu = 4\pi^{1/2} n e^4 ln\Lambda / 3M^{1/2} T^{3/2}$ with $ln\Lambda$ being the Coulomb logarithm, Catto and Simakov performed a systematic expansion of the ion kinetic equation in the small parameters δ and $\triangle$ to determine the ion distribution function to order $\delta^2 \sim \delta\triangle \sim \triangle^2$ in terms of the ion flow velocity and $q_{\|}$ and $\overrightarrow{q}_{\perp}$ above.[1] Catto and Simakov have worked out these derivations and presented here is the formulism by Catto. Simakov's formulae are essentially the same as Catto's except that they are presented in a little different forms. [4]

With a given distribution function for species j, f_j, the viscosity tensor can be found by the following formula:

$$\overleftrightarrow{\prod}_j = m_j \int d^3 w_j (\overrightarrow{w}_j \overrightarrow{w}_j - \frac{1}{3} w_j^2 \overleftrightarrow{I}) f_j.$$

With Catto and Simakov's ordering for f_j, they worked the derivation for each coordinates (parallel, perpendicular, and gyroviscous) of the tensor, which are again decomposed as we did in Braginskii's formulism:

$$\overleftrightarrow{\prod}_j = \overleftrightarrow{\prod}_{\|j} + \overleftrightarrow{\prod}_{\perp j} + \overleftrightarrow{\prod}_{gj} = \overleftrightarrow{\prod}_{\|j} + \overleftrightarrow{\prod}_{\perp j_1} + \overleftrightarrow{\prod}_{\perp j_2} + \overleftrightarrow{\prod}_{gj}.$$

For the expansion of $\overleftrightarrow{\prod}_{\perp j}$ as in $\overleftrightarrow{\prod}_{\perp j} = \overleftrightarrow{\prod}_{\perp j_1} + \overleftrightarrow{\prod}_{\perp j_2}$, the subsrcipt "1" denotes terms from the linearized collision operator, while the "2" subscript denotes the new terms that are quadratic in the heat fluxes $\overrightarrow{q}$ and $\overrightarrow{q}_{\|}$ from the nonlinear collision operator. Note that $\overleftrightarrow{\prod}_{\perp j_0} = 0$.[1]

In his paper, Catto writes $f_j = f_0 + f_1 + f + 2 \cdots$ as a sum of a gyroaveraged $\overline{f}_j$ and gyrophase dependent $\tilde{f}_j$ pieces by letting $f_j = \overline{f}_j + \tilde{f}_j$ and presents his formulism for non-zero viscosity components.[1][2]

$$\overleftrightarrow{\prod}_{\|j} = m_j^3 w (\overrightarrow{w}\,\overrightarrow{w} - \tfrac{1}{3} w^2 \overleftrightarrow{I}) \overline{f}$$

$$= \frac{0.960}{\nu} (\overleftrightarrow{I} - 3\hat{n}\hat{n}) : \left[\left(P_j \nabla \overrightarrow{V} + \frac{2}{5} \nabla \overrightarrow{q} + 0.246 (\nabla \overrightarrow{q} - \overrightarrow{q} \nabla ln P_j + \frac{4}{15} \nabla q_{\|}) \right) \right] \left(\hat{n}\hat{n} - \frac{1}{3} \overrightarrow{I} \right)$$

$$+ \frac{m_j}{P_j T_j} \left[0.412 q_{\|}^2 - 0.064 q^2 \right] \left(\hat{n}\hat{n} - \frac{1}{3} \overleftrightarrow{I} \right)$$

$$= \left(\hat{n}\hat{n} - \frac{1}{3} \overleftrightarrow{I} \right) \{ \frac{1025}{1068\nu} \left(P \nabla \cdot \overrightarrow{V} + (2/5) \nabla \cdot \overrightarrow{q} - 3P \hat{n} \cdot \nabla \overrightarrow{V} \cdot \hat{n} - (6/5) \hat{n} \cdot \nabla \overrightarrow{q} \cdot \hat{n} \right)$$

$$+ \frac{319417 m_j q_{\|}^2}{890000 TP} - \frac{21}{89\nu} \left(\overrightarrow{q} \cdot \nabla ln P - \nabla \cdot \overrightarrow{q} + 3\hat{n} \cdot \nabla \overrightarrow{q} \cdot \hat{n} - 3 \overrightarrow{q}_{\|} \cdot \nabla ln P \right)$$

$$+ \frac{28}{445\nu} \left(\nabla \cdot \overrightarrow{q}_{\|} - 3\hat{n} \cdot \nabla \overrightarrow{q}_{\|} \cdot \hat{n} \right) - \frac{1137 m_j q_{\perp}^2}{17800 TP} \}$$

$$\overleftrightarrow{\prod}_{gj} = m_j^3 w_j (\overrightarrow{w}_j \overrightarrow{w}_j - \tfrac{1}{3} w_j^2 I) \tilde{f}_2 = m_j \int d^3 w_j \overrightarrow{w}_j \overrightarrow{w}_j \tilde{f}_2$$

$$= \frac{1}{4\Omega} \{ \hat{n} \times \left[(P_j \nabla \overrightarrow{V} + \tfrac{2}{5} \nabla \overrightarrow{q}) + (P_j \nabla \overrightarrow{V}_j + \tfrac{2}{5} \nabla \overrightarrow{q})^T \right] \cdot (\overleftrightarrow{I} + 3\hat{n}\hat{n})$$

$$- (\overleftrightarrow{I} + 3\hat{n}\hat{n}) \cdot \left[(P_j \nabla \overrightarrow{V}_j + \tfrac{2}{5} \nabla \overrightarrow{q}) + (P_j \nabla \overrightarrow{V}_j + \tfrac{2}{5} \nabla \overrightarrow{q})^T \right] \times \hat{n} \}$$

$\overleftrightarrow{\prod}_{\perp 1}$
$= -\frac{3\nu}{10\Omega^2}\{P_j\nabla\overrightarrow{V}_j + \frac{2}{5}\nabla\overrightarrow{q} + \left(P_j\nabla\overrightarrow{V}_j + \frac{2}{5}\nabla\overrightarrow{q}\right)^T - \frac{3}{10P_j}[P_j\nabla\overrightarrow{q} - \overrightarrow{q}\nabla P_j + (P_j\nabla\overrightarrow{q} - \overrightarrow{q}\nabla P_j)^T] - \frac{1}{100P_j}[3P_j\nabla\overrightarrow{q}_{\parallel} + 5\overrightarrow{q}_{\parallel} + (3P\nabla\overrightarrow{q}_{\parallel} + 5\overrightarrow{q}_{\parallel}\nabla P)^T] - \frac{1}{400T}[(90\overrightarrow{q} - 13\overrightarrow{q}_{\parallel})\nabla T + (\nabla T)(90\overrightarrow{q} - 13\overrightarrow{q}_{\parallel})] + 3\hat{n}[\hat{n}\cdot(P\nabla\overrightarrow{V} + 1/10\nabla\overrightarrow{q} - 3/100\nabla\overrightarrow{q}_{\parallel}) + (P\nabla\overrightarrow{V} + 1/10\nabla\overrightarrow{q} - 3/100\nabla\overrightarrow{q}_{\parallel})\cdot\hat{n}] + 3[\hat{n}\cdot(P\nabla\overrightarrow{V} + 1/10\nabla\overrightarrow{q} - 3/100\nabla\overrightarrow{q}_{\parallel}) + (P\nabla\overrightarrow{V} + 1/10\nabla\overrightarrow{q} - 3/100\nabla\overrightarrow{q}_{\parallel})\cdot\hat{n}]\hat{n} + \frac{3q_{\parallel}}{4P}[\hat{n}\nabla P + (\nabla P)\hat{n}] + \frac{9}{10P}[\hat{n}\overrightarrow{q} + \overrightarrow{q}\hat{n} - \frac{1}{3}q_{\parallel}\hat{n}\hat{n}]\hat{n}\cdot\nabla P - \frac{231q_{\parallel}}{400T}[\hat{n}\nabla T + (\nabla T)\hat{n}] - \frac{27}{40T}[\hat{n}\nabla T + (\nabla T)\hat{n}] - \frac{27}{40T}[\hat{n}\overrightarrow{q} + \overrightarrow{q}\hat{n} - \frac{13}{45}q_{\parallel}\hat{n}\hat{n}]\hat{n}\cdot\nabla T\}$

$$\overleftrightarrow{\prod}_{\perp 2} = \frac{9m_j\nu}{200PT\Omega}[\hat{n}\times\overrightarrow{q}(\overrightarrow{q} + \tfrac{31}{15}\overrightarrow{q}_{\parallel}) - (\overrightarrow{q} + \tfrac{31}{15}\overrightarrow{q}_{\parallel})\overrightarrow{q}\times\hat{n}]$$

To complete the closure of the energy conservation equation along with $\overleftrightarrow{\prod}$ presented above, Catto presents the following froms for $\overrightarrow{q}$ and $\overrightarrow{F}$:

$$\overrightarrow{q} = (5P/2M\Omega)\hat{n}\times\nabla T - (2P\nu/M\Omega^2)\nabla_{\perp}T - (125P/32M\nu)\hat{n}\hat{n}\cdot\nabla T$$

$$\overrightarrow{F} = mn\nu_{ei}[(\overrightarrow{V}_{\perp} - \overrightarrow{V}_{\perp_e}) + 0.51(\overrightarrow{V}_{\parallel} - \overrightarrow{V}_{\parallel_e})] - (3n_{ei}/2\Omega)\hat{n}\times\nabla T_e - 0.71n\hat{n}\hat{n}\cdot\nabla T_e.$$

In Catto's $\overrightarrow{q}$ above, the collisional contribution to the perpendicular ion heat flux(the first term), which is formally smaller by ν/Ω than the order $\triangle$ parallel collisional heat flux and the order δ diamagnetic heat flux, is also added. So each terms in $\overrightarrow{q}$ above are collisional perpendicular, diamagnetic, and parallel ion heat fluxes, respectively. Catto finds $\overrightarrow{F}$ above to be the same as in Braginskii's formulism and states that higher order corrections are not required for $\overrightarrow{F}$. Catto also presents the viscosity formulism for electron species in his paper. [1]

Compared to Mikhailovskii's formulism, Catto's paralleland perpendicular viscosities given above contain additional terms to those originally presented by Mikhailovskii and Tsypin due to the need to retain the full nonlinear ion-ion collision operator. As stated earlier, some of these corrections occur because they used a truncated polynomial approximation rather than the exact gyrophase dependent portion of the ion distribution function, while the others come from the need to retain the nonlinear collision terms they neglected. The temperature gradient terms(∇T) in the perpedicular viscosity($\overleftrightarrow{\prod}_{\perp 1}$) arise from the gyrophase dependent and independent portions of the ion distribution function. [1]

The truncated Sonine polynomial expansion solution technique of Mikhailovskii and Tsypin makes two assumptions which Catto and Simakov removed to obtain completely general results. First, Mikhailovskii and Tsypin neglect contributions to the viscosity that arise from the full nonlinear form of the collision operator. This modification gives rise to heat flux squared terms($q_{\parallel}^2$, q^2, $q_{\perp}^2$, $\hat{n}\times\overrightarrow{q}(\overrightarrow{q}+\frac{31}{15}\overrightarrow{q}_{\parallel})$, $(\overrightarrow{q}+\frac{31}{15}\overrightarrow{q}_{\parallel})\overrightarrow{q}\times\hat{n}$) in the parallel($\overleftrightarrow{\prod}_{\parallel j}$) and perpendicular($\overleftrightarrow{\prod}_{\perp 2}$) viscosities that are the same size as terms found by Mikhailovskii and Tsypin. Second, because of their truncation only an approximation to the gyrophase dependent portion of the ion distribution function is retained. This approximate form is not accurate enough to completely and properly evaluate some of the terms in the perpendicular collisional viscosity. Therefore, it is possible that the modifcations to the parallel and perpendicular viscosities that Catto and Simakov present here may alter collisional and turbulent transport in some situations. [1]

3.4.1 From Catto to Braginskii

Because the viscosity formulism by Catto represents the general form, we should be able to extract Braginskii's formulim by neglecting all the temperature gradient terms (including all the heat transfer terms because heat transfer terms are functions of temperature gradients) in Catt's formulism. If we do that, we get the following simplified viscosity formulism from Catto's formulae.

$$\overleftrightarrow{\prod}_{\parallel j}=\frac{0.960}{\nu}(\overleftrightarrow{I}-3\hat{n}\hat{n}):\left[\left(P_j\nabla\overrightarrow{V}\right]\left(\hat{n}\hat{n}-\frac{1}{3}\overrightarrow{I}\right)=\left(\hat{n}\hat{n}-\frac{1}{3}\overleftrightarrow{I}\right)\left[\frac{1025}{1068\nu}[P\nabla\cdot\overrightarrow{V}-3P\hat{n}\cdot\nabla\overrightarrow{V}\cdot\hat{n}]\right]\tag{18}$$

$$\overleftrightarrow{\prod}_{gj}=\frac{1}{4\Omega}\{\hat{n}\times\left[P_j\nabla\overrightarrow{V}+(P_j\nabla\overrightarrow{V}_j)^T\right]\cdot(\overleftrightarrow{I}+3\hat{n}\hat{n})-(\overleftrightarrow{I}+3\hat{n}\hat{n})\cdot\left[P_j\nabla\overrightarrow{V}_j+(P_j\nabla\overrightarrow{V}_j)^T\right]\times\hat{n}\}\tag{19}$$

$$\overleftrightarrow{\prod}_{\perp 1}=-\frac{3\nu}{10\Omega^2}\{P_j\nabla\overrightarrow{V}_j+\left(P_j\nabla\overrightarrow{V}_j\right)^T+3\hat{n}\left[\hat{n}\cdot\left(P\nabla\overrightarrow{V}\right)+\left(P\nabla\overrightarrow{V}\right)\cdot\hat{n}\right]+3\left[\hat{n}\cdot\left(P\nabla\overrightarrow{V}\right)+\left(P\nabla\overrightarrow{V}\right)\cdot\hat{n}\right]\hat{n}\}\tag{20}$$

$$\overleftrightarrow{\prod}_{\perp 2}=0\tag{21}$$

If we rewrite these equations above, we should be able to prove that these are actually the same as the Braginskii viscosities. To do so, we can use the following relations to verify that Braginskii's results can be obtained from Catto's formulism.

$$\overleftrightarrow{\prod}_{\parallel} = \begin{pmatrix} \prod_{rr}^{0} & \prod_{r\theta}^{0} & \prod_{r\phi}^{0} \\ \prod_{\theta r}^{0} & \prod_{\theta\theta}^{0} & \prod_{\theta\phi}^{0} \\ \prod_{\phi r}^{0} & \prod_{\phi\theta}^{0} & \prod_{\phi\phi}^{0} \end{pmatrix}$$

$$\overleftrightarrow{\prod}_{\perp} = \begin{pmatrix} \prod_{rr}^{12} & \prod_{r\theta}^{12} & \prod_{r\phi}^{12} \\ \prod_{\theta r}^{12} & \prod_{\theta\theta}^{12} & \prod_{\theta\phi}^{12} \\ \prod_{\phi r}^{12} & \prod_{\phi\theta}^{12} & \prod_{\phi\phi}^{12} \end{pmatrix}$$

$$\overleftrightarrow{\prod}_{g} = \begin{pmatrix} \prod_{rr}^{34} & \prod_{r\theta}^{34} & \prod_{r\phi}^{34} \\ \prod_{\theta r}^{34} & \prod_{\theta\theta}^{34} & \prod_{\theta\phi}^{34} \\ \prod_{\phi r}^{34} & \prod_{\phi\theta}^{34} & \prod_{\phi\phi}^{34} \end{pmatrix}$$

Taking $\overleftrightarrow{\prod}_{\parallel}$ for example, the Braginskii's formulism gives the following parallel viscosity, with the superscript denoting Braginskii's formulism.

$$\overleftrightarrow{\prod}_{\parallel}^{Braginskii} = -\eta A_0 \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & -3/2f_p \\ 0 & -3/2f_p & -1 \end{pmatrix}$$

By neglecting the heat transfer terms from Catto's parallel viscosity above, we get a very similar viscosity form as given below, with the superscript denoting the reduction from Catto to Braginskii.

$$\overleftrightarrow{\prod}_{\parallel}^{Catto->Braginskii} = -\eta \left(\frac{2}{3}\frac{\partial V_r}{\partial l_r} + \frac{2}{3}\frac{\partial V_\theta}{\partial l_\theta} - 2f_p \frac{\partial V_\phi}{\partial l_\theta} \right) \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & -3/2f_p \\ 0 & -3/2f_p & -1 \end{pmatrix}$$

Additional work remains to verify that the coefficients of two expressions agree with each other when same assumptions are applied. But we can clearly see that Catto's parallel viscosity reduces to Braginskii's when the heat transfer terms are neglected. The same type of verification is also required for the other viscosities, which remains as a future work.

3.5 From Braginskii to Catto

It will also be of interest to evaluate the relative importance of each heat transfer contributions to plasma parameters. For that purpose, presented below are the collection of the heat transfer terms(and temperature gradient terms) from Catto's formulism, which we designate them with 'heat' in superscript. Therefore, these are the additional heat transfer effects being added to Braginskii formulism.

$$\overleftrightarrow{\prod}_{\parallel j}^{heat}$$
$$= \frac{0.960}{\nu}(\overleftrightarrow{I} - 3\hat{n}\hat{n}) : \left[\left(\frac{2}{5}\nabla\overrightarrow{q} + 0.246(\nabla\overrightarrow{q} - \overrightarrow{q}\nabla lnP_j + \frac{4}{15}\nabla q_{\parallel})\right)\right]\left(\hat{n}\hat{n} - \frac{1}{3}\overrightarrow{I}\right)$$
$$+ \frac{m_j}{P_j T_j}\left[0.412q_{\parallel}^2 - 0.064q^2\right]\left(\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I}\right)$$
$$= \left(\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I}\right)\{\frac{1025}{1068\nu}\left((2/5)\nabla\cdot\overrightarrow{q} - (6/5)\hat{n}\cdot\nabla\overrightarrow{q}\cdot\hat{n}\right) + \frac{319417 m_j q_{\parallel}^2}{890000TP}$$
$$- \frac{21}{89\nu}\left(\overrightarrow{q}\cdot\nabla lnP - \nabla\cdot\overrightarrow{q} + 3\hat{n}\cdot\nabla\overrightarrow{q}\cdot\hat{n} - 3\overrightarrow{q}_{\parallel}\cdot\nabla lnP\right)$$
$$+ \frac{28}{445\nu}\left(\nabla\cdot\overrightarrow{q}_{\parallel} - 3\hat{n}\cdot\nabla\overrightarrow{q}_{\parallel}\cdot\hat{n}\right) - \frac{1137 m_j q_{\perp}^2}{17800TP}\}$$

$$\overleftrightarrow{\prod}_{gj}^{heat} = \frac{1}{4\Omega}\{\hat{n}\times\left[\tfrac{2}{5}\nabla\overrightarrow{q} + (\tfrac{2}{5}\nabla\overrightarrow{q})^T\right]\cdot(\overleftrightarrow{I} + 3\hat{n}\hat{n}) - (\overleftrightarrow{I} + 3\hat{n}\hat{n})\cdot\left[\tfrac{2}{5}\nabla\overrightarrow{q} + (\tfrac{2}{5}\nabla\overrightarrow{q})^T\right]\times \hat{n}\}$$

$$\overleftrightarrow{\prod}_{\perp}^{heat}{}_1$$
$$= -\frac{3\nu}{10\Omega^2}\{\frac{2}{5}\nabla\overrightarrow{q} + \left(\frac{2}{5}\nabla\overrightarrow{q}\right)^T - \frac{3}{10P_j}[P_j\nabla\overrightarrow{q} - \overrightarrow{q}\nabla P_j + (P_j\nabla\overrightarrow{q} - \overrightarrow{q}\nabla P_j)^T] -$$
$$\frac{1}{100P_j}[3P_j\nabla\overrightarrow{q}_{\parallel} + 5\overrightarrow{q}_{\parallel} + (3P\nabla\overrightarrow{q}_{\parallel} + 5\overrightarrow{q}_{\parallel}\nabla P)^T] - \frac{1}{400T}[(90\overrightarrow{q} - 13\overrightarrow{q}_{\parallel})\nabla T +$$
$$(\nabla T)(90\overrightarrow{q} - 13\overrightarrow{q}_{\parallel})] + 3\hat{n}\left[\hat{n}\cdot\left(\frac{1}{10}\nabla\overrightarrow{q} - \frac{3}{100}\nabla\overrightarrow{q}_{\parallel}\right) + \left(\frac{1}{10}\nabla\overrightarrow{q} - \frac{3}{100}\nabla\overrightarrow{q}_{\parallel}\right)\cdot\hat{n}\right] +$$
$$3\left[\hat{n}\cdot\left(\frac{1}{10}\nabla\overrightarrow{q} - \frac{3}{100}\nabla\overrightarrow{q}_{\parallel}\right) + \left(\frac{1}{10}\nabla\overrightarrow{q} - \frac{3}{100}\nabla\overrightarrow{q}_{\parallel}\right)\cdot\hat{n}\right]\hat{n} + \frac{3q_{\parallel}}{4P}[\hat{n}\nabla P + (\nabla P)\hat{n}] +$$
$$\frac{9}{10P}[\hat{n}\overrightarrow{q} + \overrightarrow{q}\hat{n} - \tfrac{1}{3}q_{\parallel}\hat{n}\hat{n}]\hat{n}\cdot\nabla P - \frac{231q_{\parallel}}{400T}[\hat{n}\nabla T + (\nabla T)\hat{n}] - \frac{27}{40T}[\hat{n}\nabla T + (\nabla T)\hat{n}] -$$
$$\frac{27}{40T}[\hat{n}\overrightarrow{q} + \overrightarrow{q}\hat{n} - \tfrac{13}{45}q_{\parallel}\hat{n}\hat{n}]\hat{n}\cdot\nabla T\}$$

$$\overleftrightarrow{\prod}_{\perp 2}^{heat} = \overleftrightarrow{\prod}_{\perp 2} = \frac{9m_j\nu}{200PT\Omega}[\hat{n}\times\overrightarrow{q}(\overrightarrow{q} + \tfrac{31}{15}\overrightarrow{q}_{\parallel}) - (\overrightarrow{q} + \tfrac{31}{15}\overrightarrow{q}_{\parallel})\overrightarrow{q}\times\hat{n}]$$

Here we notice that $\overleftrightarrow{\prod}_{\perp 2}^{heat} = \overleftrightarrow{\prod}_{\perp 2}$, meaning that $\overleftrightarrow{\prod}_{\perp 2}$ only has non-zero value when we derive it with Catto's ordering assumptions.

4 The Flux Surface Geometry Models in solving Two-Fluid Equations for Tokamak Plasmas

4.1 The Elliptical Model and the Miller Equilibrium Model

With either form of the viscosity tensors presented in the previous section, we can solve the two-fluid euqations for plasma parameters such as the rotational velocity of tokamak plasmas. Apart from the issue of which viscosity formulism we employ, we must also consider what flux surface model we apply in numerically solving the two-fluid equations. In general, these flux surface coordinates must be determined by a numerical solution of the Grad-Shafranov equation.[5] The complexity of the equation formulism, however, does not allow simple numerical computation, especially with the complex form of the plasma flux surface shape.

Traditionally, the most popular approach was to solve it with the simple elliptical model. With this simple model, the numerical computation may be simpler but the result is not realistic due to its aggressive simplifying assumptions. A decade ago, an analytical solution of the Grad-Shafranov equation for the equilibrium magnetic flux surface geometry in tokamaks has been developed by Miller et al. in which the flux surface is completely described by the aspect ratio, elongation, triangularity and safety factor. By representing the equations of plasma physics directly within this analytical flux surface geometry, the numerical solution of the Grad-Shafranov equation step can be omitted from the calculation procedure.[5]

We can therefore describe tokamak plasmas with either of these models, the Elliptical or the Miller model. Since it is the next step for the research community to apply Braginskii formulism with the Miller model, discussions in this section will exclusively refer to Braginskii's formulism.

4.2 The Flux Surface Coordinate System

The natural coordinate system for tokamak plasma physics computations is the set of nested magnetic flux surfaces because of the striking differences in particle, momentum, and energy flows within and across these flux surfaces. In most literatures concering tokamak plasmas, the flux surface coordinate system is defined by the orthogonal coordinate directions (r, θ, ϕ) with length elements

$dl_r = h_r dr$,
$dl_\theta = h_\theta d\theta$,
and $dl_\phi = h_\phi d_\phi$.

The coordinates θ and ϕ lie in the flux surface and represent a poloidal angle-like variable θ and the toroidal angle ϕ, respectively. The r coordinate is normal to the flux surface and can be any flux surface label. [5] In this paper, we will employ this same coordinate system in our formulism.

4.3 The Elliptical Model with Braginskii formulism

The elliptical model assumes that the plasma shape is elliptical. For this model, the R and Z coordinates of the plasma are determined by the following relations:

$R = R_0 + rcos\theta$
$z = rsin\theta.$

For this model, we can use the following metric coefficients

$h_\psi = (RB_p)^{-1}$
$h_p = h_p$
$h_\phi = R = R_0 + rcos\theta = R_0(1 + \epsilon cos\theta)$

where $\epsilon = r/R_0$.

With $R = R_0(1+\epsilon cos\theta)$ and $B_\theta = B_\theta^0/(1+\epsilon cos\theta)$, the numerical application of the elliptical model becomes really simplified since $RB_\theta = R_0 B_\theta^0$ = constant. Also, in the large aspect ratio, low-β, circular flux surface approximation, we can set:

$h_\theta \approx r$
$\epsilon = r/R_0 << 1.$

These approximations make the numerical computation much simpler and was applied by many researches due to its simplicity and/or its minimal numerical computational load. Application of Braginskii's formulism with the elliptical model is well discussed in Stacey's text. [6]

4.4 The Miller Equilibrium Model with Braginskii formulism

A decade ago, Miller, et al. derived analytical expressions for an equilibrium flux surface in a plasma, which we call 'The Miller Model' in this paper. Figure 1 illustrates the Miller model which describes the flux surface with nine parameters: s (global magnetic shear), α (pressure gradient), A (aspect ratio), κ (elongation), δ (triangularity), q (safety factor), and the variation of κ, δ, and the major radius with flux surface. The positive direction of the angle θ shown in Fig. 1 corresponds to the direction of B produced by a plasma current out of the page in Fig. 1, i.e. a clockwise plasma current looking down of the tokamak.[3] [5] In the Miller model, the r coordinate is also normal to the flux and chosen such that the 'radial' displacement is $dl_r = dr/|\triangle r|$.[3]

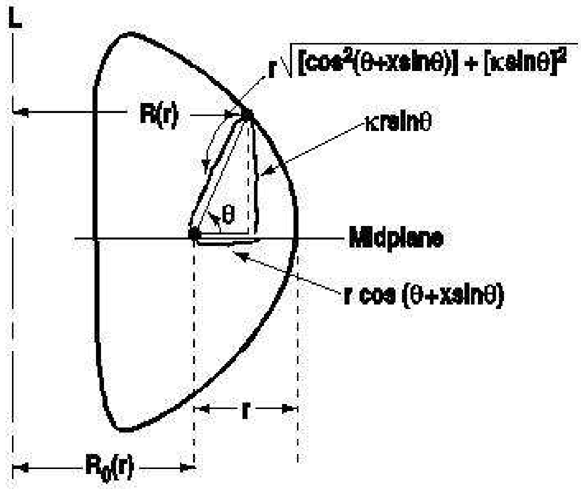

Figure 1: Miller Equilibrium Model

With the Miller model, the R and Z coordinates of the plasma are described by

$$R(r) = R_0(r) + rcos[\theta + xsin\theta] = R_0(r) + rcos\xi$$
$$Z(r) = \kappa rsin\theta$$

where $x \equiv sin^{-1}\delta$ and $\xi \equiv \theta + xsin\theta$. The Miller Model assumes $R_0 = R_0(r)$, $\delta = \delta(r)$, and $\kappa = \kappa(r)$. With this model, the metric coefficients are given as follows:

$$h_r = h_r(r,\theta) = (RB_\theta)^{-1} = \frac{1}{|\nabla(r,\theta)|}$$
$$h_\theta = h_\theta(r,\theta) = r\sqrt{cos^2(\theta + xsin\theta) + \kappa^2 sin^2\theta}$$
$$h_\phi = h_\phi(r,\theta) = R(r,\theta) = R_0(r) + rcos(\theta + xsin\theta).$$

Here we can see that these metric coefficients are of much more complicated functions of r and θ than those of the elliptical model.[3] [5]

Miller presents in his paper the detailed form of the metric coefficient as follows:

$$RB_\theta = R|\nabla\phi \times \nabla\psi| = |\nabla(r,\theta)| = \frac{\partial\psi}{\partial r}|\nabla r| = \partial_r\psi|\nabla r| = \frac{\partial\psi}{\partial r}\kappa^{-1} \bigwedge(r,\theta)$$
$$= \frac{\partial_r\psi \ \ \kappa^{-1}\sqrt{sin^2(\theta + xsin\theta)(1 + xcos\theta)^2 + \kappa^2 cos^2\theta}}{cos(xsin\theta) + \partial_r R_0 \ \ cos\theta + [s_\kappa - s_\delta cos\theta + (1 + s_\kappa)xcos\theta]sin\theta sin(\theta + xsin\theta)}$$

where

$$|\nabla r| = \kappa^{-1} \bigwedge(r,\theta) = \frac{\kappa^{-1}\sqrt{sin^2(\theta + xsin\theta)(1 + xcos\theta)^2 + \kappa^2 cos^2\theta}}{cos(xsin\theta) + \partial_r R_0 \;\; cos\theta + [s_\kappa - s_\delta cos\theta + (1 + s_\kappa)xcos\theta]sin\theta sin(\theta + xsin\theta)}$$
$$\bigwedge(r,\theta) = \frac{\sqrt{sin^2(\theta + xsin\theta)(1 + xcos\theta)^2 + \kappa^2 cos^2\theta}}{cos(xsin\theta) + \partial_r R_0 \;\; cos\theta + [s_\kappa - s_\delta cos\theta + (1 + s_\kappa)xcos\theta]sin\theta sin(\theta + xsin\theta)}.$$

s_κ and s_δ account for the change in elongation and triangularity, respectively, with radial location.

$$s_\kappa = \frac{r}{\kappa}\frac{\partial\kappa}{\partial r},$$
$$s_\delta = r\frac{\partial\delta}{\partial r}/\sqrt{(1-\delta^2)}.$$

Note that $\partial_r R_0 = \frac{\partial R_0}{\partial r}$ and $\partial_r \psi = \frac{\partial \psi}{\partial r}$. The formula of $\frac{\partial R_0}{\partial r}$ is given by two different models. The shifted circle model (which leads to the Sahfranov shift) yields

$$\frac{\partial R_0}{\partial} \equiv \triangle' = -\frac{r}{R_0}\left(\beta_\theta + \frac{1}{2}\right)$$

and a shifted ellipse model by Lao, et al. yields

$$\frac{\partial R_0}{\partial r} = -\frac{r}{R_0}\left[\frac{2(\kappa^2+1)}{(3\kappa^2+1)}\left(\beta_\theta + \frac{1}{2}l_i\right) + \frac{1}{2}\frac{(\kappa^2-1)}{(3\kappa^2+1)}\right].$$

Here $\beta_\theta = \frac{nT}{B_\theta^2/2\mu_0}$ and l_i is the internal inductance. The definition of the safety factor $q(r) = \frac{|B_{\phi 0}|}{2\pi}\int \frac{dl_\theta}{RB_\theta}$ and RB_θ equation can be used to evaluate $\frac{\partial\psi(r)}{\partial r} = \frac{|B_{\phi o}|\kappa(r)}{2\pi q(r)}\int \frac{dl_\theta}{[1 + \frac{r}{R_0(r)}cos(\theta + xsin\theta)]\Lambda(r,\theta)}$. [3] [5]

Though this new model by Miller was available for a decade, it has not been employeed in solving the two-fluid equations to compute tokamak plasma parameters such as the plasma rotation velocities. This was due to its increased numerical computation loads to computers in the past and/or its complexity in application. It will, however, be interesting to apply this Miller model in our calculation of plasma parameters of modern tokamaks.

5 Conclusion

In solving the two-fluid equations of magnetized plasmas, either Braginskii's or Mikhilovskii's viscosity formulism can be employeed. The decision on which formulism fits best for our research will depend on the physics of the magnetized plasma. For example, if the ion flow velocity is weaker than the ion thermal speed and on the order of the ion diamagnetic heat flux divided by the pressure, the Mikhailovskii's viscosity approach must be used. In other words, Catto and Simakov's formulism will provide the best computational result since they have evolved out of Mikhailovskii's approach with more terms in truncating the Sonine polinomials. If, however, the flow velocity is almost equal to the ion thermal speed, Braginskii's formulism will be adequate enough because using Catto and Simakov's formulism will increase the amount of numerical computational work dramatically.

Another important aspect in solving the two-fluid equation is in which flux surface geotmetry model to employ. Among many models available in the research community, the most popular one up to date has been the Elliptical Model due to its minimal computational load and/or its simplicity, but lacking the detailed flux surface parameters for more accurate computational results. Therefore it will be interesting to apply the new model, the Miller Equilibrium Model, in solving the two-fluid equations since it describes the realistic D-shaped tokamak flux surfaces with more parameters.

With two types of viscosity formulisms and two types of flux surface models, we may choose any combination to numerically describe tokamak plasmas. Braginskii's formulism in the elliptical model was worked out by Stacey and well presented in his book. [6] It will be next logical step to apply Braginskii's formulism with the Miller model.

References

[1] P. J. Catto and Simakov A. N. A drift ordered short mean free path description for magnetized plasma allowing strong spatial anisotropy. *Physics of Plasma*, 11(1), 2004.

[2] P. J. Catto and A. N. Simakov. Evaluation of the neoclassical radial electric field in a collisional tokamak. *Physics of Plasmas*, 12(012501), 2005.

[3] Chu M. S. Greene J. M. Lin-Liu Y. R. Miller, R. L. and R. E. Waltz. Noncircular, finite aspect ratio, local equilibrium model. *Physics of Plasmas*, 5(4), 1998.

[4] A. N. Simakov and P. J. Catto. What are the limitations of braginskii's fluid equations and hazeltine's drift kinetic equation? *Theory of Fusion Plasmas: Joint Varenna-Lausanne International Workshop*, pages 238–249, 2006.

[5] W. M. Stacey and C. Bae. Representation of the plasma fluid equations in 'miller equilibrium' analytical flux surface geometry. , May 2009.

[6] Weston M. Stacey. *Fusion Plasma Physics*. Wiley-VCH, 2005.