1) **Huffman codes**

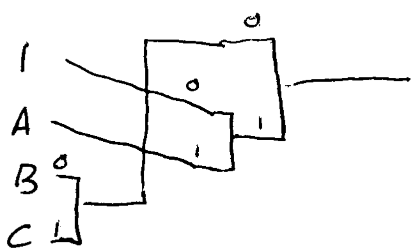
Consider a source that outputs one of *m* IID symbols with probabilities, $p_1 > p_2 \geq \ldots \geq p_m$.

a) Prove that for any binary Huffman code, if the largest probability is $p_1 > 2/5$, then the codeword assigned to this symbol is of length 1.

b) Prove that for any binary Huffman code, if the smallest probability is $p_m < 1/3$, then the codeword assigned to this symbol is at least length 2.

Contradiction

a) Assume $P_1 > 2/5$ has length 2 → then the following must have occurred in H-tree:



Note: $P_1 > 2/5$, $P_B + P_C > 2/5$ } $P_A < 1/5$ ← contradiction

But $P_A \geq P_B$ & $P_A \geq P_C$

$\Rightarrow P_A + P_A \geq P_B + P_C \Rightarrow$

$P_A \geq \frac{P_B + P_C}{2} > \frac{2/5}{2} > \frac{1}{5}$

Contradiction

b) Note 2 smallest probs have same length, L_s for H.C.

Assume $L_s = 1$, see if this satisfies K.I.

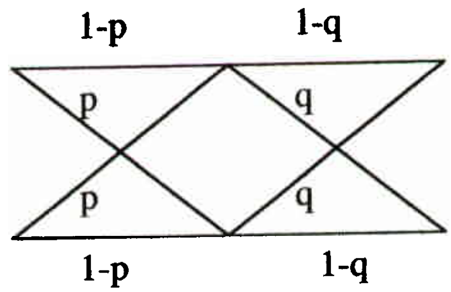
Kraft inequality $\sum_{i=1}^{m} 2^{-l_i} \leq 1$

$$\sum_{i=1}^{m-2} 2^{-l_i} + 2^{-1} + 2^{-1} \leq 1$$

$$\Rightarrow \sum_{i=1}^{m-2} 2^{-l_i} \leq 0 \quad \Leftarrow \text{not possible}$$

2) Capacity of cascaded BSCs.

a) Compute the capacity and find the maximizing input distribution for this cascaded channel.



b) Suppose that the order of the channels was reversed, namely the BSC with crossover probability *q* was first. Assuming that p<q, which channel ordering gives higher capacity? Justify your answer.

a) $P_p = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$, $P_q = \begin{bmatrix} 1-q & q \\ q & 1-q \end{bmatrix}$

$P = P_p P_q = \begin{bmatrix} (1-p)(1-q)+pq & (1-p)q+(1-q)p \\ (1-q)p+(1-p)q & (1-p)(1-q)+pq \end{bmatrix}$

$C = 1 - H(\underline{r})$ $\quad \underline{r}$ row of P (channel is symmetric)

$p(x=0) = p(x=1) = 1/2$

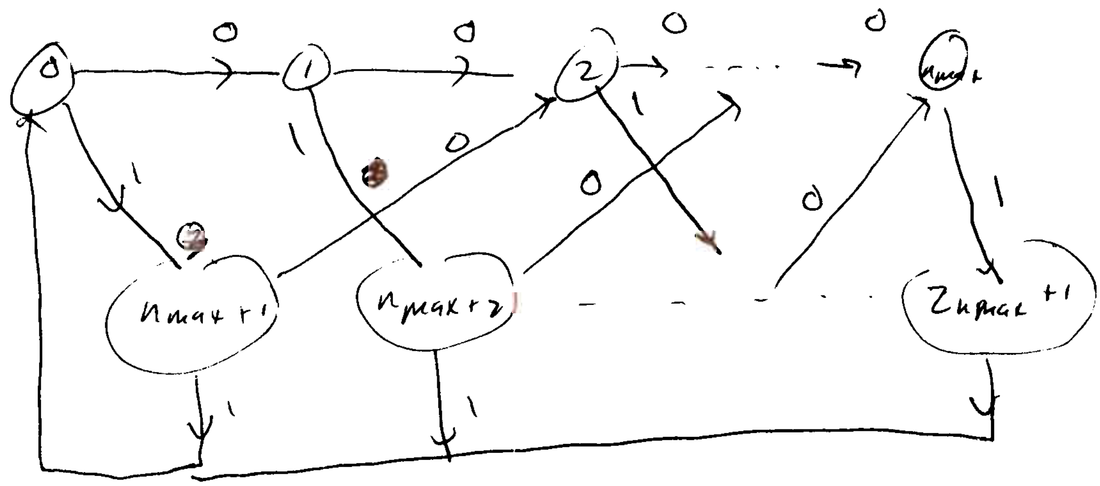
b) Capacity is same for both channel ordering.

3) **Capacity of the 'one-pairs' constraint.** Consider a binary channel that places a constraint on **the maximum run, n_{max}, of bits between pairs of ones**. For n_{max}=3 the following are examples of allowable sequences { 111, 110011, 1100011, 1101011}, but the following are not allowable sequences { 11000011, 11010011}.

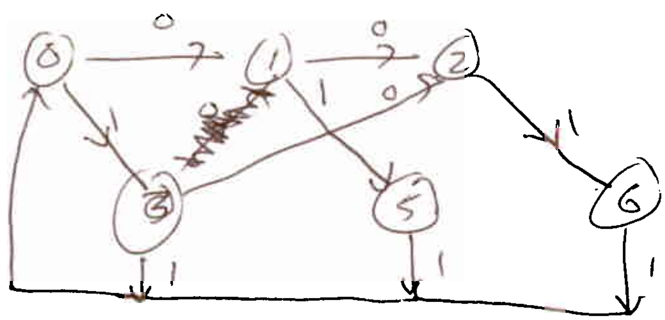
a) give a general constraint graph (where the state transitions are labeled with only one bit) that describes the n_{max} **one-pairs** constraint

b) give the constraint graph for n_{max}=2 compute its capacity

a)



b)



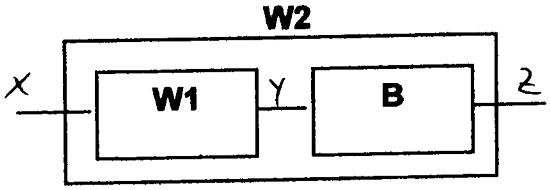
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C = \log_2 \lambda$

λ is largest eigenvalue of A

4) In class, we have seen that there exists a stochastic matrix associated with every discrete memoryless channel (DMC). For example for an **m** input **n** output DMC, let **W** be an **n-by-m** matrix with that contains the transition probabilities as its entries and the sum of each column is one. We have seen that the convex hull of the columns of **W** is very important, since every output distribution lies in the convex hull **conv W**. More precisely, every point within **conv W** is a distribution on the output and can be written as a convex combination of the columns of **conv W.**

A. Consider two channels $\mathbf{W_1}$ and $\mathbf{W_2}$ and suppose that **conv** $\mathbf{W_2}$ is contained in **conv** $\mathbf{W_1}$. Show that $\mathbf{W_2 = W_1 B}$ where B is a (stochastic) matrix representing a second channel that is cascaded with $\mathbf{W_1}$ to produce $\mathbf{W_2}$ as shown in the following figure:



B. For the same assumptions of part A, prove that the mutual information of $\mathbf{W_2}$ is smaller than or equal to mutual information of $\mathbf{W_1}$ for every input distribution. As a result we say $\mathbf{W_1}$ is a better channel than $\mathbf{W_2}$ and denote it by $\mathbf{W_2 \preceq W_1}$, if **conv** $\mathbf{W_2}$ is contained in **conv** $\mathbf{W_1}$.

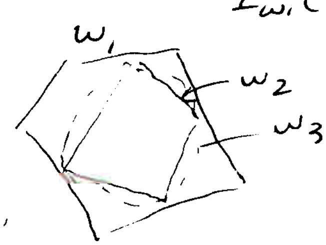
C. Consider a DMC channel which is represented with stochastic matrix $\mathbf{W_3 = pW_1 + (1-p)W_2}$, where $0<p<1$. If $\mathbf{W_1}$ is better then $\mathbf{W_2}$, prove that $\mathbf{W_3 \preceq W_1}$.

a) Note $w2$ is a stochastic matrix
$w1$ " " "

$\therefore$ $B = [b_1 \cdots b_2] \Rightarrow w_2 = w_1 B$

b) All you need is data processing theorem $\Rightarrow I_{w_2}(X;Z) \le I_{w_1}(X;Y)$

c) Note: $\text{Conv}\, w_3 = \text{conv}\, w_1 - \text{conv}\, w_2$



w_3 is conv hull but ocpcl
$\Rightarrow$ anywhere between w_1 and w_2
$\Rightarrow w_3 \le w_1$ because $C_1 \ge C_3$