Connecting spectral techniques for graph coloring and Eigen properties of coupled dynamics: A pathway for solving combinatorial optimizations (Invited Paper)

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***Abstract*—This paper reviews an analog circuit system of capacitively coupled relaxation oscillators whose time evolution can be used to solve the graph coloring problem. These oscillators consist of a series combination of an insulator-metal-transition (IMT) device and a resistance. Such circuits were also demonstrated experimentally using VO_2 (Vanadium Dioxide) as the phase transition material. The time evolution of circuit dynamics depend on eigenvectors of the adjacency matrix in the same way as is used by spectral algorithms for graph coloring. As such, a coupled network of such oscillators with piecewise linear dynamics have steady state phases which can be used to approximate the minimum vertex coloring of a graph.**

I. Introduction

Many of the hard computational problems in the present world fall in the category of discrete, or combinatorial, optimization problems. These are the problems where an optimal value of a function, or its optimal point, needs to computed within a domain set which is discrete, e.g. the set of integers, or combinatorial, e.g. all permutations of some set. Currently we use a platform-algorithm approach of solving most problems which uses a platform (turing machine) and provide it with a set of instructions (algorithm) to be executed. Problems are *easy* or *hard* depending on whether an intelligent algorithm can be found which is not simply a brute force comparison of function values on the whole domain set. What makes such problems hard is not only the characteristic of the problem, but also the limitations of the computing platform, like (a) single computing node that executes instructions, (b) the discrete sequential nature of executing instructions, (c) separation of memory and logic, among others. On the contrary, computing in nature replaces the discrete sequential character of computing with a continuous one, the processes occur in parallel, and memory is distributed along with operations. Creating such a nature-inspired computing system essentially means creating a system which evolves in continuous time and can be described using continuous time differential equations. But realizing a physical system which obeys any arbitrary set of differential equations is hard. In case of designing electrical systems, for instance, using passive elements like resistors and capacitors limits the realizability to low order linear dynamical systems. This poses a challenge as the dynamical systems which can solve NP-hard problems like SAT are known to be very non-linear [2].

In [1] we create such a continuous time dynamical system for solving the Vertex Coloring problem, or Graph Coloring problem, which is one of the most studied NP-hard combinatorial optimization problem not only for its significance in computational theory but also for its many real world applications like fault diagnosis [3], scheduling [4]–[6], resource allocation [7], and many more. The objective of *vertex k-coloring*, or *graph k-coloring*, is to assign one color (out of total k colors) to each vertex of a graph such that no two adjacent vertices receive the same color. The minimum k for which a correct coloring is possible is called the *chromatic number* of the graph. Finding a correct k-coloring, or even approximating it, is known to be hard. [8], [9].

In [1] we describe a circuit composed of coupled relaxation oscillators which have piecewise linear charging and discharging dynamics [10]. We describe a system that uses such a circuit as the first step, and then process its output to approximate the *minimum vertex coloring* problem. Computing using these oscillators is phase based, i.e. the information is encoded in the phase of oscillators. Two such identical relaxation oscillators when coupled using a capacitance tend to synchronize in an anti-phase locked condition [10]. Such anti-phase locking is analogous to the oscillators "pushing" each other in phase, and any in-phase locking synchronization would corresponds to a "pulling" effect and as such could perform coloring. The formal explanation of this effect can be presented using the spectral properties of the couple doscillator dynamical system and spectral algorithms for graph coloring.

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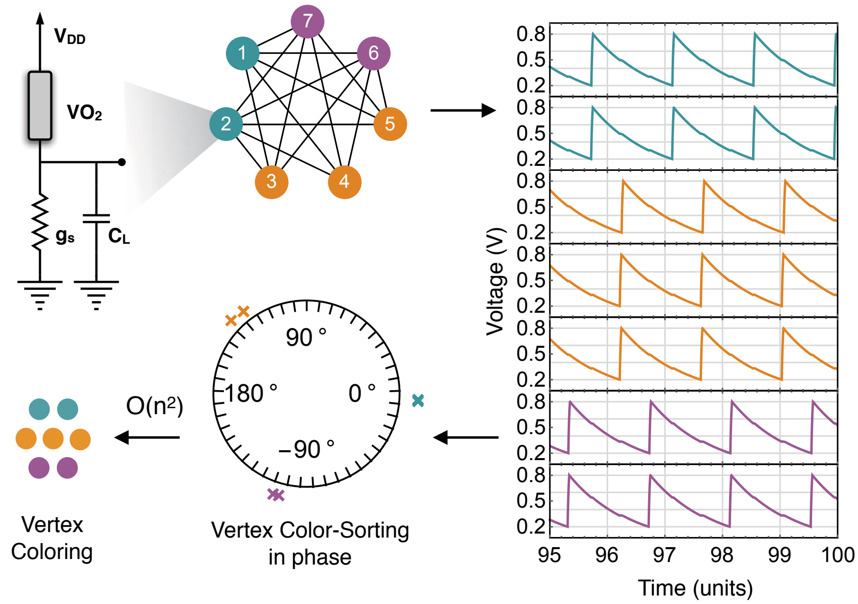

Fig. 1. System overview for solving the graph coloring optimization problem. The coupled relaxation oscillator circuit is used to generate a phase ordering of the oscillators in steady state, which is then used to assign colors to the nodes, which is a $O(n^2)$ operation. (Inset) The inner circuit for a VO_2 relaxation oscillator, which is a VO_2 device is series with a conductance g_s and a loading capacitor C_L in parallel.

II. Work

A. Circuit Description

The working of such oscillators depend on the state transition property of the IMT devices. These devices switch between a low resistance metal state with conductance g_m and a high resistance insulator state with conductance g_i based on the voltage v_d across their two terminals. On increasing v_d an IMT device switches to a metallic state (insulator-to-metal (IMT) transition) after a threshold v_h, and on decreasing v_d below v_l the device switches to an insulating state (metal-to-insulator (IMT) transition). Here $v_h > v_l$ and there is hysteresis in switching. When voltage is applied across the series combination of an IMT device and a conductance g_s, initially at $t = 0$ the IMT device is in high resistance state and the voltage drop across the device $v_d = 0$. The internal capacitance of the device charges up and v_d increases and eventually crosses the threshold v_h. This makes the device metallic which causes the internal capacitance of the device to discharge and reduces v_d which finally drops below v_l resulting in oscillations with piecewise linear dynamics. An extra loading capacitance is added to the circuit as shown in figure 1.

The dynamics for a single oscillator can be written as the following piecewise differential equation:

$$cv'(t) = -g(s)v(t) + p(s)$$

where c is the lumped capacitance of device along with the parasitics, s is the state of system - charging (1) or discharging (0), and $g(s)$ is the net path conductance in state s, and $g(s) = g_s + g_i s$. If the voltage v is normalized to v_{dd} then $p(s) = s$. Such identical oscillators when coupled capacitively tend to synchronize in anti-phase locking condition [10]. The dynamics of a circuit of identical coupled relaxation oscillators can be described using the following matrix equation:

$$v'(t) = (C_i + C_c + C_l)^{-1} \left[-G(S)v(t) + g_i S\right]$$

Here, v is the vector of all voltages, C_i is a diagonal matrix with the diagonal elements equal to the internal capacitances of the corresponding oscillator nodes, C_c is the coupling capacitance matrix with diagonal elements equal to the degree of that node with positive sign and off-diagonal elements with negative sign, S is the vector of states of all oscillators, $G(S)$ is a state dependent diagonal matrix with $diag(G(S)) = g_s + g_i S$, and I is the identity matrix. C_l is a diagonal matrix corresponding to the extra loading capacitors. These loading capacitors effectively add to the internal capacitance, and are chosen such that $diag(C_c + C_l)$ is constant. When all oscillators have equal internal capacitances c_i and the coupling capacitances are identical and equal to c_c, then $C_i = c_i I$ and $C_c = c_c L$, L being the laplacian matrix of the connection graph and $L = D - A$ where D is a diagonal matrix of degrees of vertices and A is the adjacency matrix of the graph. In such a case, a simple choice of C_l is $C_l = c_c(nI - D)$. We design these oscillators for very high charging and low discharging rates. As such, for most of the time, a coupled oscillator system is in a state where all oscillators are discharging, i.e. $S = 0$ (figure 1). The system starts with an initial permutation determined by $v(0)$, and finally settles down to a limit cycle with some final permutation encoded as the order of charging spikes of oscillators. The phases of charging spikes are represented as a polar plot in figure 1. For most graphs, the final order of charging spikes gives the solution of *vertex color-sorting* problem with high probability.

B. Continuous Formulation for vertex coloring

The *vertex coloring* problem can be recast to a continuous domain optimization by changing the representation of the coloring function we seek. In an integer programming formulation of graph coloring, an integer color assignment $\psi(v_i)$ is sought for each vertex v_i. In approximation algorithms using semidefinite programming relaxation and spectral algorithms, there is an intermediate assignment $\psi'(v_i)$ which maps vertices to vectors (vector coloring) and $\psi(v_i)$ is calculated by clustering $\psi'(v_i)$ and eventually mapping the clusters to discrete colors. In our case, to understand the dynamics of coupled relaxation oscillators and their ability to color graphs, the intermediate function $\psi'(v_i)$ maps vertices to real numbers (which are phases of oscillators) such that the permutation of vertices that sorts $\psi'(v_i)$ in increasing or decreasing order is such that same colored vertices occur together, i.e. if vertices u and v should have the same color and if there is any vertex z such that $\psi'(u) \leq \psi'(z) \leq \psi'(v)$ then z should also have the same color as u and v. We call this problem *vertex color-sorting*. It can be shown easily that calculating ψ from a correct permutation ψ' is $O(n^2)$ [1], and hence hardness of the original *vertex coloring* problem is maintained in *vertex color-sorting*. In the relaxation oscillator circuit, the value $\psi'(v_i)$ is given by the phase of oscillator corresponding to v_i. The phases of oscillators are unique only upto a reference phase, and as the phases are 2π periodic the permutation that sorts the phases is a cyclic permutation, i.e. unique upto a cyclic rotation.

C. Dynamics of permutation search

The system usually has a limit cycle when the charging rates are much higher than the discharging rates. An analysis of two such coupled oscillators was done in [10]. In such case, most of time the system is in the state $S = 0$. Also we see that order of charging transitions should have a strong correlation to the order of components of $v(t)$ in the discharging state $S = 0$. Hence, we try to study separately the linear dynamics of system in $S = 0$ state, and the fast charging spikes that finally become periodic.

First, we just consider the linear dynamical system in the state $S = 0$ without any charging transitions, and understand what is the asymptotic order of components of $v(t)$ in such a system. Any ordering P of the components $v_{i1} > v_{i2} > ... > v_{in}$ corresponds to a region in $\mathbb{R}^n$ which is a pair of simplexes, both having one vertex as the origin and are mirror images of each other [1]. Let's call this a *permutation region* $\mathcal{R}_{\mathcal{P}}(P)$ corresponding to the order P. Any line through the origin either lies in both these simplexes or none. Hence, the asymptotic order of components of $v(t)$ depends on the permutation region in which the asymptotic direction of the system state lies. If $d(v_0)$ is the asymptotic direction if the system starts from an initial point $v(0) = v_0$, then

$$d(v_0) = \lim_{t \to \infty} \frac{v(t)}{\|v(t)\|}$$

In the state $S = 0$, the system equation is simply

$$v'(t) = -g_s \left(cI + c_c L + C_l\right)^{-1} v(t)$$

which is an autonomous linear dynamical system with a fixed point 0. Due to addition of C_l, the coefficient matrix $-g_s \left(cI + c_c L + C_l\right)^{-1}$ is of the form $g_s \left(c_c A - \alpha I\right)^{-1}$ where A is the adjacency matrix of the connection graph. The coefficient matrix is symmetric, and hence its eigenvalues are real and we can find orthonormal eigenvectors. Moreover, all eigenvalues are negative and the eigenvectors are same as that of A [1]. As such, the system is attracted towards the fixed point and asymptotically becomes tangential to the eigenspaces. If $\mu_1 > \mu_2 > \cdots > \mu_l$ are the distinct eigenvalues of coefficient matrix and $E_1, E_2, \cdots, E_l$ their corresponding eigenspaces ($l \leq n$), then the asymptotic direction $d(v_0) \in E_1$. Specifically, $d(v_0) = P_{E_1} v_0$ where P_{E_1} is the projection matrix on the subspace E_1 [1]. In case $d(v_0)$ lies at the boundary of some permutation regions, i.e. any two or more components of $d(v_0)$ are equal (or close), then the asymptotic order of the corresponding components is decided by the dynamics in residual projection space $E_2 \oplus E_3 \oplus \cdots \oplus E_l$. This reduced dynamics in this space is equivalent to the dynamics starting at initial point $v(0) = (I - P_{E_1}) v_0$ and the coefficient matrix projected onto the residual space by pre-multiplying with $(I - P_{E_1})$. Let the asymptotic direction of system state in this space be $d(v_0 \backslash E_1)$, then

$$\begin{aligned} d(v_0 \backslash E_1) &= P_{E_2} v_0 \\ d(v_0 \backslash E_1 \oplus E_2) &= P_{E_3} v_0 \end{aligned}$$

and so on [1]. The asymptotic order of the components of $v(t)$ in the linear dynamical system of state $S = 0$ without any transitions is decided by all these asymptotic directions in the subspaces until the subspace E such that $d(v_0 \backslash E)$ is not close to any boundary of the permutation regions. Figure 2 shows such an ordering in a representative 2-dimensional system. It should be noted that E_1 corresponds to the least negative eigenvalue μ_1 of the coefficient matrix, but corresponds to the most negative eigenvalue of the adjacency matrix A.

D. Results and Relation to spectral algorithms

The reason why eigenspaces corresponding to most negative eigenvalues direct the system towards a correct vertex coloring can be found in the analysis of spectral algorithms for graph coloring [11]–[13] where it has been shown that the adjacency matrix eigenvectors with (most) negative eigenvalues can be used to color graphs. Aspvall and Gilbert [13] show that for strongly regular graphs, the eigenvectors with negative eigenvalues have components which are constant on color classes and starting from the most negative eigenvalues, these components can be used to find the color classes. Alon and Kahale [12] proved that for a 3-partite graph with equal class

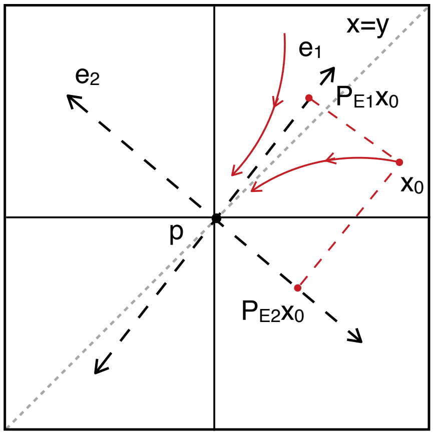

Fig. 2. Representative figure showing the relation between the asymptotic order of components of the state vector and the eigenspaces in a two-dimensional linear system

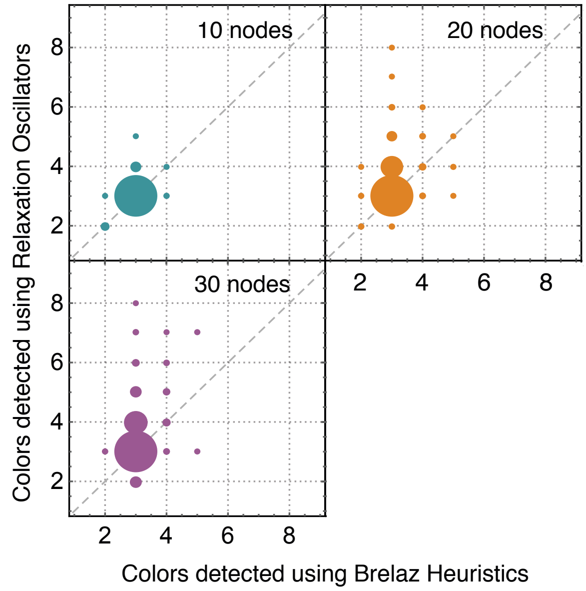

Fig. 3. Number of colors detected using the relaxation oscillator circuit plotted against number of colors detected using Brelaz heuristics for the random graph instances. The size of dots represent the percentage of instances for which the pair of colors detected using the two methods lie at the center of corresponding dot

sizes and the number of nodes $n \to \infty$, the components of two adjacency matrix eigenvectors with the most negative eigenvalues are almost constant on every color class. In the same paper, it was also proved that the last two eigenvalues are separated from the rest. In context of analysis presented in section II-C, these results imply that in 3-partite graphs with equal class sizes and $n \to \infty$, or 3-partite graphs which are strongly regular, if e_n and e_{n-1} are the last two eigenvectors, then the asymptotic direction $d(v_0 \backslash e_n \oplus e_{n-1})$ should have an order such that the components corresponding to different color classes are separated. The remaining eigenvectors/eigenspaces merely decide the order within color classes which is of no significance for vertex coloring. General graphs can be considered as perturbations from these specific cases and the system gives good approximation for their coloring.

Figure 3 shows a comparison of number of colors detected using our coupled oscillator system with that using Brelaz Heuristics for many random graphs generated using a random graph model which are all 3 colorable. Table I shows another such comparison for DIMACS graph instances [14], comparing the colors detected using Brelaz heuristics and those using relaxation oscillator circuit starting from 4 random initial states. As we see, the coupled oscillator system is able to color most graphs with comparable performance as the heuristics.

TABLE I
COMPARISON OF DIMACS GRAPHS

Graph	Number of Nodes	Chromatic Number χ	Brelaz Heuristics	Coupled Oscillator
huck	74	11	11	12,12,12,12
myciel3	11	4	4	4,4,4,4
myciel4	20	5	5	5,5,5,5
myciel5	47	6	6	8,6,6,7
myciel6	95	7	7	10,11,8,8
david	87	11	11	15,13,13,13
queen5_5	25	5	7	8,9,6,9
queen6_6	36	7	10	13,12,12,12
queen7_7	49	7	12	14,12,13,14
queen8_8	64	9	15	14,16,15,16
DSJC125 .1	125			
DSJC125 .5	125			
DSJC125 .9	125			

III. CONCLUSION

IMT based coupled relaxation oscillator system is a very interesting example of device-architecture-algorithm codesign showing a practical way of solving combinatorial optimization using an electrical circuit based dynamical system. More importantly, it highlights a fundamental connection between spectral algorithms and linear dynamical systems, both of which are governed by the spectral properties of the connectivity matrix. Further, such a system by construction embodies many of the characteristics of new architectures being explored like distributed computation and memory, and parallel evaluation among many others. Such networks can pave the way for efficient realization of algorithmic and architectural principles in the form of dynamical systems.

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