Spring 2001, Aaron Lanterman

ECE 7251: Signal Detection and Estimation
Some Problems on MMSE Estimation

1 Problems

1. This problem and the next one were made up by Bruce Hajek of UIUC; I used them as a homework problems when I taught ECE434, the grad probability and random processes class, at UIUC. The solutions to these two problems were LaTexed by Cheng Tang who was my T.A.

   Let random variables X and Y have joint density

   $$f(x,y) = \begin{cases} x+y & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the *linear* MMSE estimator of X given Y.

   (b) What is the mean-square error of the LMMSE you computed in part (a)?

   (c) Find the (possibly nonlinear) MMSE estimator of X given Y.

   (d) What is the mean-square error of the MMSE you computed in part (c)?

   (e) Compute the ratio of the mean-square errors, i.e., divide your answer in part (d) (the MMSE squared-error) by your answer in part (b) (the LMMSE error).

2. Repeat parts (a) through (e) of the previous problem for the case where Y is a $N(0,1)$ random variable and $X = |Y|$.

3. This is another homework problem by Bruce Hajek. He also used it on one of his old final exams. I gave it on a midterm exam.

   Define a random process $(X_t, t \in \Re)$ by $X_t = A + Bt + t^2$, where A and B are independent $N(0,1)$ random variables.

   (a) (10 pt) Find $\hat{E}[X_5|X_1]$, the *linear* minimum mean square error (LMMSE) estimator of X_5 given X_1.

   (b) (10 pt) Compute the mean square error of the LMMSE estimator.

   (c) (7 pt) Find the (possibly nonlinear) minimum mean square error estimator of X_5 given X_1.

   (d) (8 pt) Find $\hat{E}[X_t|X_1, X_0]$, and compute its mean square error.

2 Solutions

1. (a) The linear MMSE estimator is given by

$$\hat{E}[X|Y] = E[X] + \text{Cov}(X,Y)\text{Cov}(Y,Y)^{-1}(y - E[Y]) \quad (1)$$

which gives

$$\hat{E}[X|Y] = \frac{7}{12} - \frac{1}{11}(Y - \frac{7}{12}). \quad (2)$$

One can also get the same result using the orthogonality principle. Let $\hat{E}[X|Y] = aY + b$, then solving the following equation

$$E[X - (aY + b)] = 0, \; E[(X - (aY + b))Y] = 0$$

will give the linear MMSE as in (2).

(b) $E[e_\ell^2] = E[(X - \hat{E}[X|Y])^2] = \text{Cov}(X,X) - \text{Cov}(X,Y)^2/\text{Var}(Y) = 5/66 \approx 0.0757575$

(c) Since we know the joint density $f(x,y)$, we have

$$f_Y(y) = \int_0^1 (x+y)dx = \frac{1}{2} + y, \; f_{X|Y}(x|y) = \frac{x+y}{\frac{1}{2}+y}.$$

This gives the best MMSE estimator

$$E[X|Y] = \int_0^1 x f_{X|Y}(x|y)dx = \frac{\frac{1}{3} + \frac{1}{2}Y}{\frac{1}{2} + Y}.$$

(d) $E[e^2] = E[(X - E[X|Y])^2] = E[X^2] - E[(E[X|Y])^2] = \frac{5}{12} - [\frac{1}{3} + \frac{\ln 3}{144}] = \frac{12 - \ln 3}{144} \approx 0.7570408$

(e) The MMSE gives 99.93% of the MSE from the linear estimator.

2. (a) Given $E[X] = \frac{2}{\sqrt{2\pi}}, E[Y] = 0, \text{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0, \text{Var}(Y) = 1$ and using (1), one has the linear MMSE estimator

$$\hat{E}[X|Y] = \frac{2}{\sqrt{2\pi}} + \frac{0}{1}Y = \sqrt{\frac{2}{\pi}}.$$

(b) $E[e_\ell^2] = E[(X - \hat{E}[X|Y])^2] = E[X^2] - (\frac{2}{\sqrt{2\pi}})^2 = 1 - \frac{2}{\pi}.$

(c) $E[X|Y] = E[|Y||Y] = |Y|$. Thus the MMSE estimator is exactly the same as X!

(d) $E[e^2] = 0$.

(e) The best nonlinear estimator is perfect in the sense of MSE! It has 100% smaller MSE than the linear estimator.

3. (a) Find $\hat{E}[X_5|X_1]$, the *linear* minimum mean square error (LMMSE) estimator of X_5 given X_1.
Solution:

$$\begin{aligned} X_5 &= A + 5B + 25 \\ X_1 &= A + B + 1 \end{aligned}$$

$E[X_5] = 25$, $E[X_1] = 1$, and

$$Cov([X_1 \quad X_5]^T) = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 26 & 6 \\ 6 & 2 \end{bmatrix}$$

$$\hat{E}[X_5|X_1] = E[X_5] + \frac{Cov[X_1, X_5]}{Var[X_1]}(X_1 - E[X_1]) = 25 + \frac{6}{2}(X_1 - 1) = 22 + 3X_1$$

(b) Compute the mean square error of the LMMSE estimator.

$$MSE = Var[X_5] - \frac{(Cov[X_1, X_t])^2}{Var[X_1]} = 26 - \frac{36}{2} = 26 - 18 = 8$$

(c) Find the (possibly nonlinear) minimum mean square error estimator of X_5 given X_1.
Solution: Since X_t is Gaussian, $\hat{E}[X_5|X_1] = E[X_5|X_1]$.

(d) Find $\hat{E}[X_t|X_1, X_0]$, and compute its mean square error.
Solution: $X_0 = A$, and $X_1 = A + B + 1$, so $B = X_1 - A - 1 = X_1 - X_0 - 1$. Hence

$$X_t = X_0 + (X_1 - X_0 - 1)t + t^2$$

This is clearly a linear estimator. Fixing X_1 and X_0 completely determines X_t for all t, so the MSE is zero.