ECE 7251: Signal Detection and Estimation

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Lecture 24, 3/11/01:

Chernoff Bounds (Gaussian Examples)

Material from Last Lecture

- Consider the *log*likelihood ratio test

$$L \equiv \ln \Lambda = \ln \frac{p(y\,|\,H_1)}{p(y\,|\,H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln l \equiv g$$

- Main object of interest: $m(s) \equiv \ln \Phi_{L|H_0}(s)$

$$\Phi_{L|H_0}(s) \equiv E[e^{sL} \,|\, H_0] = \int_{-\infty}^{\infty} e^{sl}\, p(l\,|\,H_0)\,dl$$

$$= \int_{y} p(y\,|\,H_1)^s\, p(y\,|\,H_0)^{1-s}\, dy$$

- Both representations will be useful
- Discussion based on Van Trees, pp. 126-129

Ex. 1: Gaussian, Equal Variances

$$H_1 \sim N(m, s^2),\; H_0 \sim N(0, s^2)$$

$$m(s) = \ln \int_{y} p(y\,|\,H_1)^s\, p(y\,|\,H_0)^{1-s}\, dy$$

$$= \ln \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \prod_{i=1}^{n} \frac{1}{\sqrt{2ps^2}} \exp\left[\frac{(y_i - m)^2}{2s^2} \right] \right\}^s \times \left\{ \prod_{i=1}^{n} \frac{1}{\sqrt{2ps^2}} \exp\left[\frac{y_i^2}{2s^2} \right] \right\}^{1-s} dy_1 \cdots dy_n$$

$$= n \ln \int_{-\infty}^{\infty} \frac{1}{\sqrt{2ps^2}} \exp\left[\frac{(y-m)^2 s + y^2(1-s)}{2s^2} \right] dy$$

Ex. 1: Completing the Square

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2ps^2}} \exp\left[\frac{(y-m)^2 s + y^2(1-s)}{2s^2} \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2ps^2}} \exp\left[-\frac{(y^2 - 2my + m^2)s + y^2(1-s)}{2s^2} \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2ps^2}} \exp\left[-\frac{y^2 - 2msy + m^2 s + m^2 s^2 - m^2 s^2}{2s^2} \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2ps^2}} \exp\left[-\frac{(y^2 - 2msy + m^2 s^2) - m^2 s^2 + m^2 s}{2s^2} \right] dy$$

Ex. 1: Finish Computing the Mu

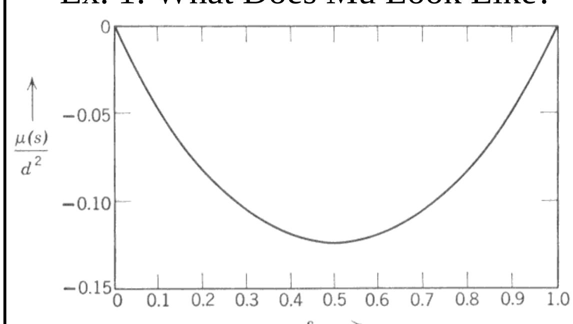
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2ps^2}} \exp\left[-\frac{(y^2 - 2msy + m^2 s^2) - m^2 s^2 + m^2 s}{2s^2} \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2ps^2}} \exp\left[-\frac{(y - ms)^2}{2s^2} \right] \exp\left[-\frac{m^2 s(1-s)}{2s^2} \right] dy$$

$$= \exp\left[\frac{m^2 s(s-1)}{2s^2} \right]$$

$$m(s) = n \ln\{\downarrow\} = \frac{s(s-1)}{2} \frac{nm^2}{s^2} \equiv \frac{s(s-1)}{2} d^2$$

Ex. 1: What Does Mu Look Like?



(Graph from p. 126 of Van Trees Vol. I)

Ex. 1: Basic Bound on P_{FA}

$$m(s) = \frac{s(s-1)}{2} d^2, \quad \dot{m}(s) = \frac{2s-1}{2} d^2$$

$$P_{FA} \le \exp[m(s) - s\dot{m}(s)] \quad \text{for } 0 \le s \le 1$$

$$= \exp\left[\frac{s(s-1)}{2} d^2 - s\frac{(2s-1)}{2} d^2\right] = \exp\left[-\frac{s^2}{2} d^2\right]$$

where $g = \dot{m}(s)$ $\qquad g = \frac{2s-1}{2} d^2$

$$s = \frac{g}{d^2} + \frac{1}{2}$$

Ex. 1: Basic Bound on P_m

$$P_m \le \exp[m(s) + (1-s)\dot{m}(s)]$$

$$= \exp\left[\frac{s(s-1)}{2} d^2 + (1-s)\frac{(2s-1)}{2} d^2\right]$$

$$= \exp\left[\frac{s^2 - s}{2} d^2 + \frac{2s - 1 - 2s^2 + s}{2} d^2\right]$$

$$= \exp\left[\frac{2s - 1 - s^2}{2} d^2\right] = \exp\left[-\frac{(1-s)^2}{2} d^2\right]$$

Ex. 1: Where are the Bounds Meaningful?

- Recall we need

$$E[L \mid H_0] \le g \le E[L \mid H_1]$$

$$\dot{m}(0) \le g \le \dot{m}(1)$$

$$\frac{2 \cdot 0 - 1}{2} d^2 \le g \le \frac{2 \cdot 1 - 1}{2} d^2$$

$$-\frac{d^2}{2} \le g \le \frac{d^2}{2}$$

Ex. 1: The Refined Bound for P_{FA}

- Recall the refined asymptotic bound:

$$P_{FA} \approx \exp[m(s) - s\dot{m}(s)] \exp\left[\frac{s^2 \ddot{m}(s)}{2}\right] Q(s\sqrt{\ddot{m}(s)})$$

$$\dot{m}(s) = \frac{2s-1}{2} d^2, \quad \ddot{m}(s) = d^2$$

- In this case, since L is a sum of Gaussian random variables, the expression is exact:

$$P_{FA} = \exp\left[-\frac{s^2}{2} d^2\right] \exp\left[\frac{s^2 d^2}{2}\right] Q(sd) = Q(sd)$$

Ex. 1: The Refined Bound for P_M

$$P_M \approx e^{m(s) + (1-s)\dot{m}(s)} \exp\left[\frac{(s-1)^2 \ddot{m}(s)}{2}\right] Q((1-s)\sqrt{\ddot{m}(s)})$$

$$= \exp\left[-\frac{(1-s)^2}{2} d^2\right] \exp\left[\frac{(s-1)^2 d^2}{2}\right] Q((1-s)d)$$

- Again, since L is Gaussian, the expression is exact:

$$P_M = Q((1-s)d)$$

Exercise: Using MATLAB, plot basic Chernoff bound and true probs. of false alarm on the same graph vs. *d* for various choices of *g*

Ex. 1: Minimum Prob. of Error

- For minimum prob. of error test, $g = 0$

$$s_M = \frac{g}{d^2} + \frac{1}{2} = \frac{1}{2}$$

- Recall approximate expression for P_e from last slide of last lecture

$$P_e \approx \frac{1}{2s_m(1-s_m)\sqrt{2p\ddot{m}(s_m)}} \exp[m(s_m)]$$

$$= \frac{1}{2s_m(1-s_m)\sqrt{2pd^2}} \exp\left[\frac{s_m(s_m-1)}{2} d^2\right]$$

Ex. 1: Min. Prob. of Error Con't

$$P_e \approx \frac{2}{\sqrt{2p d^2}} \exp\left[-\frac{d^2}{8}\right]$$

- Recall the exact expression is:

$$P_e = Q(d/2)$$

- Van Trees' rule of thumb: "approximation is very good for *d*>6"

The Bhattacharyya Distance

- If the criterion is the minimum prob. of error and *m(s)* is symmetric about *s*=1/2, then

$$m(s) = \ln \int_y \sqrt{p(y\,|\,H_1)}\sqrt{p(y\,|\,H_0)}\,dy$$

- *-m(s)* is called the Battacharyya distance

Ex. 2: Gaussian, Equal Means

$$H_1 \sim N(0, s_1^2),\; H_0 \sim N(0, s_0^2)$$

Exercise: Veryify that

$$m(s) = \frac{n}{2} \ln \frac{(s_0^2)^s (s_1^2)^{1-s}}{s s_0^2 + (1-s) s_1^2}$$

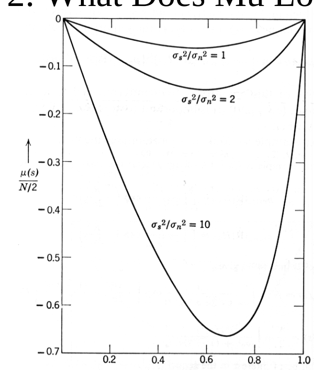
- A common special case:

$$s_1^2 = s_s^2 + s_n^2,\; s_0^2 = s_n^2$$

Exercise: Veryify that

$$m(s) = \frac{n}{2}\left\{(1-s)\ln\left[1+\frac{s_s^2}{s_n^2}\right] - \ln\left[1+(1-s)\frac{s_s^2}{s_n^2}\right]\right\}$$

Ex. 2: What Does Mu Look Like?



(Graph from p. 128 of Van Trees Vol. I)

Ex. 2: Gaussian, Equal Means

Exercise: Veryify that

$$\dot{m}(s) = \frac{n}{2}\left\{-\ln\left[1+\frac{s_s^2}{s_n^2}\right] + \frac{s_s^2/s_n^2}{1+(1-s)s_s^2/s_n^2}\right\}$$

$$\ddot{m}(s) = \frac{n}{2}\left\{\frac{s_s^2/s_n^2}{1+(1-s)s_s^2/s_n^2}\right\}^2$$

Exercise: Find an expression for the *s* which gives the tightest bound in terms of *g*

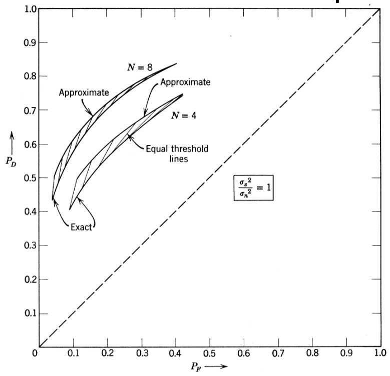
Ex. 2: Gaussian, Equal Means

Exercise: Using MATLAB, plot four different ROC curves on the same graph:

1) The true curve, computed using expressions from Lecture 21 and built in MATLAB commands like chi2pdf, etc.
2) The curve given by the basic Chernoff upper bound (this gives an upper bound ROC curve)
3) The curve given by the asymptotic Chernoff expression using the *Q* functions
4) The curve given by the asymptotic Chernoff expression using the approximation to *Q*

Try it for different values of *n* and s_s^2 / s_n^2

Ex. 2: ROC Curve Comparison



(Graph from p. 129 of Van Trees Vol. I)