ECE 7251: Signal Detection and Estimation

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Lecture 26, 3/15/01:

Uniformly Most Powerful Tests

An Introductory Case

- Usual parametric data model $p(y;q)$
- Consider a composite problem:

$$H_0 : q = q_0,\ H_1 : q \in \mathcal{S}_1$$

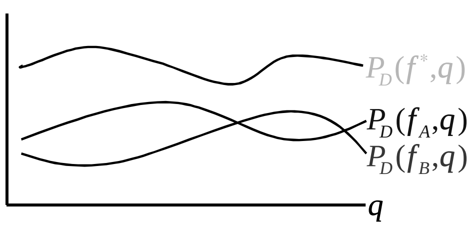
- A test f^* is <u>unformly most powerful</u> <u>of level</u> a=P_{FA} if it has a better P_D (or at least as good as) than any other a–level test

$$P_D(f^*;q) = E_q[f^*] \geq E_q[f] = P_D(f,q)$$
$$\text{for all } q \in \mathcal{S}_1, \text{ for all } f$$

- Hero uses notation b instead of P_D

Graphical Interpretation

$$a = P_{FA}(f_B) = P_{FA}(f_A) = P_{FA}(f^*)$$



Finding UMP Tests (When They Exist)

- Find the most powerful a-level (recall a=P_{FA}) test for a fixed q
  - Just the Neyman-Pearson test
- If the decision regions do not vary with q, then the test if UMP

Gaussian Mean Example

- Suppose we have n i.i.d samples

$$y_i \sim \mathcal{N}(m, s^2)$$

- Assume s^2 is known, but m is not
- Consider three cases

$$H_0 : m = 0$$

Case I: $H_1 : m > 0$

Case II: $H_1 : m < 0$

Case III: $H_1 : m \neq 0$

Sufficies to use

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\bar{y} \sim \mathcal{N}(m, s^2/n)$$

The Gaussian Likelihood Ratio

$$\Lambda(y;m) = \frac{p(y;m)}{p(y;0)} = \frac{\exp[-(\bar{y}-m)^2/(2s^2/n)]}{\exp[-\bar{y}^2/(2s^2/n)]}$$

$$= \frac{\exp[(-\bar{y}^2 + 2\bar{y}m - m^2)/(2s^2/n)]}{\exp[-\bar{y}^2/(2s^2/n)]}$$

$$= \exp\left[\frac{nm}{s^2}\bar{y} - \frac{nm^2}{2s^2}\right] \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$\frac{\sqrt{n}m}{s}\bar{y} \underset{H_0}{\overset{H_1}{\gtrless}} \left(\ln t + \frac{nm^2}{2s^2}\right)\frac{s}{\sqrt{n}} \equiv g$$

Case I: *m*>0

$$\frac{\sqrt{n}m}{s}\bar{y} \underset{H_0}{\overset{H_1}{\gtrless}} g \longrightarrow \frac{\sqrt{n}}{s}\bar{y} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{g}{m} = g^{+}$$

- Set the threshold to get the right "level"

$$a = P_{FA} = \Pr\left[\frac{\sqrt{n}}{s}\bar{y} > g^{+} \mid H_0\right] = Q(g^{+})$$

$$g^{+} = Q^{-1}(a)$$

- Notice the test does not depend on *m*; hence, it is UMP

Power of the Single-Sided Test (Case I)

$$P_D = \Pr\left[\frac{\sqrt{n}}{s}\bar{y} > g^{+} \mid H_1\right]$$

$$= \Pr\left[\frac{\sqrt{n}}{s}(\bar{y}-m) > g^{+} - m\frac{\sqrt{n}}{s} \mid H_1\right]$$

$$= Q\left(g^{+} - \frac{\sqrt{n}}{s}m\right)$$

$$= Q\left(Q^{-1}(a) - \frac{\sqrt{n}}{s}m\right) \equiv Q(Q^{-1}(a) - d)$$

Case II: *m*<0

$$\frac{\sqrt{n}m}{s}\bar{y} \underset{H_0}{\overset{H_1}{\gtrless}} g \longrightarrow \frac{\sqrt{n}}{s}\bar{y} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{g}{m} = g^{-}$$

Notice flip!

$$a = P_{FA} = \Pr\left[\frac{n}{s}\bar{y} < g^{-} \mid H_0\right] = 1 - Q(g^{-})$$

$$g^{-} = Q^{-1}(1-a)$$

- Case II is UMP also

Power of the Single-Sided Test (Case II)

$$P_D = \Pr\left[\frac{\sqrt{n}}{s}\bar{y} < g^{-} \mid H_1\right]$$

$$= \Pr\left[\frac{\sqrt{n}}{s}(\bar{y}-m) < g^{-} - m\frac{\sqrt{n}}{s} \mid H_1\right]$$

$$= 1 - Q\left(g^{-} - \frac{\sqrt{n}}{s}m\right) = 1 - Q\left(Q^{-1}(1-a) - \frac{\sqrt{n}}{s}m\right)$$

$$= 1 - Q(Q^{-1}(1-a) - d)$$

Case III: *m*≠0

$$\frac{\sqrt{n}m}{s}\bar{y} \underset{H_0}{\overset{H_1}{\gtrless}} g$$

- Uh oh… we can't just absorb *m* into the threshold anymore without effecting the inequalities!
- Decision region varies with sign of *m*
- No UMP test exists

Cauchy Median Example

- Suppose we have a single sample from the density

$$p(y;q) = \frac{1}{p}\frac{1}{1+(y-q)^2}$$

and we want to decide $H_0 : q = 0,\ H_1 : q > 0$

- Likelihood ratio is

$$\frac{p(y;q)}{p(y;0)} = \frac{1+y^2}{1+(y-q)^2} \underset{H_0}{\overset{H_1}{\gtrless}} t$$

- Decision region depends on θ, so no UMP exists!

The Monotone Likelihood Ratio Condition

$$H_0 : q = q_0,\ H_1 : q \in \mathcal{S}_1$$

- Suppose we have a Fisher Factorization

$$p(y;q) = g(T(y),q)h(y)$$

- A UMP test of any level *a* exists if the likelihood ratio is either monotone increasing or decreasing in *T* for all $q \in \mathcal{S}_1$

$$\Lambda(y;q) = \frac{p(y;q)}{p(y;q_0)} = \frac{g(T,q)}{g(T,q_0)} \equiv \Lambda(T;q)$$

(abuse)

Densities Satisfying Monotone Likelihood Ratio Condition

- Suppose we have a one sided test:

$$H_0 : q = q_0,\ H_1 : q > q_0$$
$$or\ \ H_0 : q = q_0,\ H_1 : q < q_0$$

- The following satisfy the MLR condition:
  - i.i.d. samples from 1-D exponential family (Gaussian, Bernoulli, Exponential, Poisson, Gamma, Beta)
  - i.i.d. samples from uniform density U(0,*q*)
  - i.i.d. samples from shifted Laplace

also works if $H_0 : q < q_0$

Densities *Not* Satisfying Monotone Likelihood Ratio Condition

- Gaussian with single sided H_1 on mean but *unknown* variance
- Cauchy density with single sided H_1 on centrality parameter
- Exponential family with double sided H_1