ECE 7251: Signal Detection and Estimation

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Lecture 27, 3/20/02:
Locally Most Powerful Tests

One-Sided LMP Tests

- Usual parametric data model $p(y;q)$
- Consider a single sided problem:

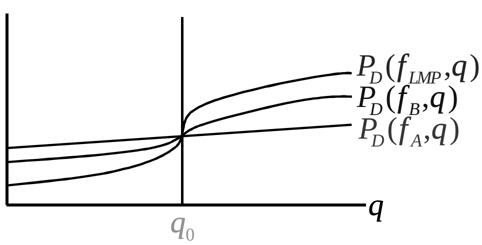
$$H_0 : q = q_0,\ H_1 : q > q_0$$

- What to do if there is no UMP test?
- The locally most powerful test of level a has a power curve which maximizes the slope of $P_D(q)$ at $q = q_0$

$$f_{LMP} = \arg\max_{f \in \{a-level\}} \frac{d}{dq} P_D(f;q) = \arg\max_{f \in \{a-level\}} \frac{d}{dq} E_q[f]$$

Graphical Interpretation of LMP Test

$$a = P_{FA}(f_B) = P_{FA}(f_A) = P_{FA}(f_{LMP})$$



Solution to LMP Problem

- Using a proof similar to that for the Neyman-Pearson lemma, one can show the LMP test is

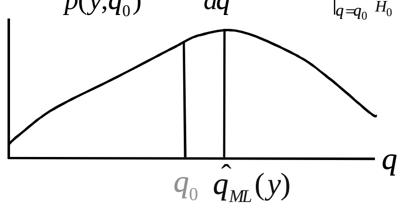
$$\frac{\frac{d}{dq} p(y;q)\Big|_{q=q_0}}{p(y;q_0)} \underset{H_0}{\overset{H_1}{\gtrless}} l$$

where we pick l to achieve $E_{q_0}[f] \le a$

- As before, we may need a randomized test if there is a nonzero prob. of landing exactly on the threshold (but we won't worry about that)

Another Interpretation of LMP Test

$$\frac{\frac{d}{dq} p(y;q)\Big|_{q=q_0}}{p(y;q_0)} = \frac{d}{dq} \ln p(y;q)\Big|_{q=q_0} \underset{H_0}{\overset{H_1}{\gtrless}} l$$



Decide H_0 if the slope at q_0 is negative or q_0 is near a stationary point of the loglikelihood, i.e. if q_0 is near the ML estimate of q

One-Sided Gaussian Mean Example

$$y_i \in \mathcal{N}(q, s^2),\ H_0 : q = 0,\ H_1 : q > 0$$

$$\frac{d}{dq} \ln p(y;q) = \frac{d}{dq}\left[c - \sum_{i=1}^{n} \frac{(y-q)^2}{2s^2}\right]\Bigg|_{q=0} \underset{H_0}{\overset{H_1}{\gtrless}} l$$

$$\frac{n\bar{y}}{s^2} = \sum_{i=1}^{n} \frac{y_i - q}{s^2}\Bigg|_{q=0} \underset{H_0}{\overset{H_1}{\gtrless}} l$$

$$\underbrace{\frac{\sqrt{n}}{s}\bar{y} \underset{H_0}{\overset{H_1}{\gtrless}} l \frac{s}{\sqrt{n}}}_{} = g$$

Just the UMP test we discussed last lecture

Cauchy Median Example

- Suppose we have n Cauchy samples:

$$p(y;q)=\prod_{i=1}^{n}\frac{1}{p}\frac{1}{1+(y_i-q)^2}$$

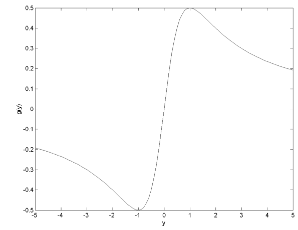
and we want to decide $H_0: q=0,\ H_1: q>0$

$$\frac{d}{dq}\ln p(y;q)=\frac{d}{dq}\{c-\ln[1+(y_i-q)^2]\}$$

$$2\sum_{i=1}^{n}\frac{(y_i-q)}{1+(y_i-q)^2}\Bigg|_{q=0}=2\sum_{i=1}^{n}\frac{y_i}{1+y_i^2}\underset{H_0}{\overset{H_1}{\gtrless}} l$$

Cauchy Median Example Con't

- Step 1: Pass each data point through a memoryless nonlinearity $g(y)=y/(1+y^2)$



- Step 2: Sum all the nonlinearity outputs
- Step 3: Compare to a threshold

Two-Sided LMP Tests

- Now consider a double sided problem:

$$H_0: q=q_0,\ H_1: q\neq q_0$$

- The locally most powerful "unbiased" test of level a has a power curve which maximizes the *curvature* of $P_D(q)$ at $q=q_0$

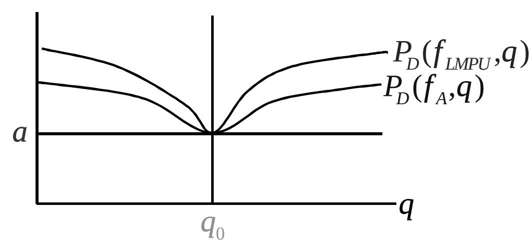
$$f_{LMP}=\arg\max_{f}\frac{d^2}{dq^2}P_D(f;q)\Bigg|_{q=q_0}$$

subject to $P_{FA}(f)=E_{q_0}[f]\leq a$

$$\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}q}P_D(f,q)\Bigg|_{q=q_0}=\frac{\mathrm{d}}{\mathrm{d}q}E_q[f]\Bigg|_{q=q_0}=0$$

Graphical Interpretation of LMPU Test

$$a=P_{FA}(f_A)=P_{FA}(f_{LMPU})$$



- Say a test is unbiased if $P_D(q)>P_{FA}$ for all q

Solution to LMPU Problem

- Using Lagrange multiplers, one can show that the test is

$$\frac{d^2}{dq^2}p(y;q)\Bigg|_{q=q_0}\underset{H_0}{\overset{H_1}{\gtrless}} l\left(p(y;q_0)+r\frac{d}{dq}p(y;q)\Bigg|_{q=q_0}\right)$$

where l and r are picked to satisfy constraints

- If we're really lucky, it sometimes turns out that $r=0$

$$\frac{\dfrac{d^2}{dq^2}p(y;q)\Big|_{q=q_0}}{p(y;q_0)}\underset{H_0}{\overset{H_1}{\gtrless}} l$$

Two-Sided Gaussian Mean Example

$$y_i\in\mathcal{N}(q,s^2),\ H_0: q=0,\ H_1: q\neq 0$$

$$p(y;q)=\frac{1}{\sqrt{2ps^2/n}}\exp\left[-\sum_{i=1}^{n}\frac{(\bar{y}-q)^2}{2s^2/n}\right]$$

$$\frac{d}{dq}p(y;q)=\frac{\bar{y}-q}{s^2/n}p(y;q)$$

$$\frac{d^2}{dq^2}p(y;q)=\frac{\bar{y}-q}{s^2/n}\left[\frac{d}{dq}p(y;q)\right]-\frac{1}{s^2/n}p(y;q)$$

$$=\left[\left(\frac{\bar{y}-q}{s^2/n}\right)^2-\frac{1}{s^2/n}\right]p(y;q)$$

Two-Sided Gaussian Example Con't

$$\frac{d^2}{dq^2} p(y;q)\Big|_{q=q_0} \underset{H_0}{\overset{H_1}{\gtrless}} l\left(p(y;q) + r\frac{d}{dq} p(y;q)\Big|_{q=q_0} \right)$$

$$\frac{\left[\left(\frac{\bar{y}}{s^2/n}\right)^2 - \frac{1}{s^2/n}\right] p(y;0)}{\left(p(y;0) + r\,\frac{\bar{y}}{s^2/n}\, p(y;0)\Big|_{q=q_0} \right)} \underset{H_0}{\overset{H_1}{\gtrless}} l$$

That Pesky Roh Lagrange Multiplier

$$\frac{\bar{y}^2 - s^2/n}{\left(s^2/n + r\bar{y}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} l(s^2/n)$$

- Usual first attempt: Wishful thinking! Try $r = 0$, and hope it works!
- Assuming this works for the moment, the test is

$$|\bar{y}| \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{l(s^2/n)^2 + s^2/n} \equiv g$$

Setting the Threshold

$$a = P_{FA} = \Pr[|\bar{y}| > g \mid H_0] = 1 - \Pr[-\bar{y} < g < \bar{y} \mid H_0]$$

$$= 1 - \Pr\left[-g\frac{\sqrt{n}}{s} < \underbrace{\bar{y}\frac{\sqrt{n}}{s}}_{\mathcal{N}(0,1)} < g\frac{\sqrt{n}}{s} \,\Big|\, H_0 \right]$$

$$= 2Q(g\sqrt{n}/s)$$

$$g = \frac{s}{\sqrt{n}} Q^{-1}(a/2)$$

Computing the Power Curve

$$P_D(q) = \Pr[|\bar{y}| > g \mid H_1] = 1 - \Pr[-g < \bar{y} < g \mid H_1]$$

$$= 1 - \Pr\left[-(g+q)\frac{\sqrt{n}}{s} < \underbrace{(\bar{y}-q)\frac{\sqrt{n}}{s}}_{\mathcal{N}(0,1)} < (g-q)\frac{\sqrt{n}}{s} \,\Big|\, H_0 \right]$$

$$= Q\left(-(g+q)(\sqrt{n}/s)\right) + 1 - Q\left((g-q)(\sqrt{n}/s)\right)$$

$$= Q\left(-(Q^{-1}(a/2)+d)\right) + 1 - Q\left(Q^{-1}(a/2)-d\right)$$

$$= 2 - Q\left(Q^{-1}(a/2)+d\right) - Q\left(Q^{-1}(a/2)-d\right)$$

where $d = q\sqrt{n}/s$