Prof. Hero's Book

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3.5 Exercises (page 29)

2. (a) $$P_\theta(\underline{x}|\underline{s}) = P_\theta(\underline{X}=\underline{x}|\underline{S}=\underline{s}) = \frac{P_\theta(\underline{S}=\underline{s}|\underline{X}=\underline{x})\,P_\theta(\underline{x})}{\sum_{\text{all } \underline{x}} P_\theta(\underline{S}=\underline{s}|\underline{X}=\underline{x})\,P_\theta(\underline{x})}$$

$$P_\theta(\underline{S}=\underline{s}|\underline{X}=\underline{x}) = \begin{cases} 1, & \text{if } \underline{s}=\underline{S}(\underline{x}) \\ 0, & \text{if } \underline{s}\neq\underline{S}(\underline{x}) \end{cases}$$

$$\therefore \sum_{\text{all } \underline{x}} P_\theta(\underline{S}=\underline{s}|\underline{X}=\underline{x})\,P_\theta(\underline{x}) = \sum_{\{\underline{x}:\, \underline{s}=\underline{S}(\underline{x})\}} P_\theta(\underline{x})$$

$$\therefore P_\theta(\underline{x}|\underline{s}) = \frac{P_\theta(\underline{S}=\underline{s}|\underline{X}=\underline{x})\,P_\theta(\underline{x})}{\sum_{\{\underline{x}:\, \underline{s}=\underline{S}(\underline{x})\}} P_\theta(\underline{x})}$$

(c) if $\underline{s}=\underline{S}(\underline{x}) \Rightarrow$ $$P_\theta(\underline{x}|\underline{s}) = \frac{P_\theta(\underline{x})}{\sum_{\{\underline{x}:\, \underline{s}=\underline{S}(\underline{x})\}} P_\theta(\underline{x})} = \frac{g(\underline{S},\theta)\,h(\underline{x})}{\sum_{\{\underline{x}:\, \underline{s}=\underline{S}(\underline{x})\}} g(\underline{S},\theta)\,h(\underline{x})}$$

$$= \frac{\cancel{g(\underline{s},\theta)}\,h(\underline{x})}{\cancel{g(\underline{s},\theta)}\sum_{\{\underline{x}:\, \underline{s}=\underline{S}(\underline{x})\}} h(\underline{x})}$$

$$\therefore P_\theta(\underline{x}|\underline{s}) = \begin{cases} \dfrac{h(\underline{x})}{\sum_{\{\underline{x}:\, \underline{s}=\underline{S}(\underline{x})\}} h(\underline{x})}, & \underline{S}(\underline{x})=\underline{s} \\ 0, & \text{otherwise} \end{cases}$$

3. $$P_\lambda(x) = \frac{\lambda^x}{x!}e^{-\lambda} = \frac{1}{x!}\exp\{x\ln\lambda - \lambda\} \longrightarrow \text{exponential family}$$

$$P_\lambda(\underline{x}) = \prod_{i=1}^n P_\lambda(x_i) = \left\{\prod_{i=1}^n \frac{1}{x_i!}\right\}\exp\Big[(\ln\lambda)\underbrace{\sum_{i=1}^n x_i}_{T} - n\lambda\Big]$$

sufficient statistic : $T = \sum_{i=1}^n x_i$

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4. $X_i \sim \text{Uniform}(\theta_1, \theta_2)$

define $I_{(a,b)}(x) = \begin{cases} 1, & x \in (a,b) \\ 0, & x \notin (a,b) \end{cases}$ $\qquad P_{X_i}(x_i) = \frac{1}{\theta_2 - \theta_1} I_{(\theta_1, \theta_2)}(x_i)$

$$P_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\theta_2 - \theta_1} I_{(\theta_1, \theta_2)}(x_i) = \frac{1}{(\theta_2 - \theta_1)^n} \prod_{i=1}^{n} I_{(\theta_1, \theta_2)}(x_i)$$

$$= \underbrace{\frac{1}{(\theta_2 - \theta_1)^n} I_{(\theta_1, \theta_2)}(\underbrace{\min\{x_i\}}_{S_1}) \, I_{(\theta_1, \theta_2)}(\underbrace{\max\{x_i\}}_{S_2})}_{g([S_1, S_2], \theta_1, \theta_2)}$$

sufficient statistic : $\underline{S}(\underline{X}) = \left[\min_i \{X_i\}, \quad \max_i \{X_i\} \right]$

5. $\theta = \beta$

(a) $$P_\theta(\underline{z}) = \prod_{i=1}^{n} P_\theta(z_i) = \left[\frac{\beta}{\sqrt{2\pi}\,\Phi(\alpha)}\right]^n \left\{\prod_{i=1}^{n}\frac{1}{z_i^2}\right\} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\left(\alpha - \frac{\beta}{z_i}\right)^2\right\}$$

$$= \underbrace{\left[\frac{\beta}{\sqrt{2\pi}\,\Phi(\alpha)}\, e^{-\alpha^2/2}\right]^n}_{a(\theta)} \underbrace{\left\{\prod_{i=1}^{n}\frac{1}{z_i^2}\right\}}_{b(\underline{z})} \exp\left\{\underbrace{\alpha\beta}_{c_1(\theta)} \underbrace{\sum_{i=1}^{n}\frac{1}{z_i}}_{d_1(\underline{z})} - \underbrace{\frac{\beta^2}{2}}_{c_2(\theta)} \underbrace{\sum_{i=1}^{n}\frac{1}{z_i^2}}_{d_2(\underline{z})}\right\}$$

$\therefore$ $P_\theta(\underline{z})$ is a member of exponential family of distributions

(b) $$P_\theta(\underline{z}) = \underbrace{\left[\frac{\beta}{\sqrt{2\pi}\,\Phi(\alpha)}\, e^{-\alpha^2/2}\right]^n \exp\left\{\left[\alpha\beta,\ -\frac{\beta^2}{2}\right]\left[\sum_{i=1}^{n}\frac{1}{z_i},\ \sum_{i=1}^{n}\frac{1}{z_i^2}\right]^T\right\}}_{g(T,\theta)} \underbrace{\prod_{i=1}^{n}\frac{1}{z_i^2}}_{h(\underline{z})}$$

sufficient statistic: $T = \left[\sum_{i=1}^{n}\frac{1}{z_i},\ \sum_{i=1}^{n}\frac{1}{z_i^2}\right]$

when $\alpha = 0 \Rightarrow P_\theta(\underline{z}) = \underbrace{\left(\sqrt{\frac{2}{\pi}}\,\beta\right)^n \exp\left\{-\frac{\beta^2}{2}\sum_{i=1}^{n}\frac{1}{z_i^2}\right\}}_{g(T,\underline{z})} \underbrace{\prod_{i=1}^{n}\frac{1}{z_i^2}}_{h(\underline{z})}$

sufficient statistic: $T = \sum_{i=1}^{n}\frac{1}{z_i^2}$

6. (a) $$f(\underline{x} \mid \bar{X} = \bar{x}; \mu) = \frac{f(\bar{x} \mid \underline{x}; \mu)\, f(\underline{x}; \mu)}{f(\bar{x}; \mu)}$$

$$f(\bar{x} \mid \underline{x}; \mu) = \begin{cases} 1, & \text{if } \bar{x} = 1/n \sum_{i=1}^{n} x_i \\ 0, & \text{if } \bar{x} \neq 1/n \sum_{i=1}^{n} x_i \end{cases}$$

$$f_{X_i}(x_i; \mu) = \frac{1}{\mu\sqrt{2\pi}}\, e^{-(x_i-\mu)^2/2\mu^2} \Rightarrow f_{\underline{X}}(\underline{x}; \mu) = \prod_{i=1}^{n} f_{X_i}(x_i; \mu)$$

$$f_{\underline{X}}(\underline{x};\mu) = \left(\frac{1}{\mu\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\mu^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$= \left(\frac{1}{\mu\sqrt{2\pi e}}\right)^n \exp\left\{-\frac{1}{2\mu^2}\sum_{i=1}^{n}x_i^2 + \frac{1}{\mu}\sum_{i=1}^{n}x_i\right\}$$

Since X_i's are iid and each has Gaussian distribution, $\bar{X}$ is also a Gaussian distribution: $E(\bar{x}) = 1/n \sum_{i=1}^{n} E(X_i) = \mu$

$$Var(X_i) = E(X_1^2) - E(X_i)^2 \Rightarrow E(X_i^2) = Var(X_i) + E(X_i)^2 = \mu^2 + \mu^2 = 2\mu^2$$

$$E(\bar{x}^2) = \frac{1}{n^2}\left\{\sum_{i=1}^{n}E(X_i^2) + \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\underbrace{E(X_i X_j)}_{E(X_i)E(X_j)}\right\}$$

$$= \frac{1}{n^2}\left\{2n\mu^2 + (n^2-n)\mu^2\right\} = \left(1+\frac{1}{n}\right)\mu^2$$

$$Var(\bar{x}) = E(\bar{x}^2) - E(\bar{x})^2 = \left(1+\frac{1}{n}\right)\mu^2 - \mu^2 = \mu^2/n$$

$$\therefore f_{\bar{X}}(\bar{x};\mu) = \frac{1}{\sqrt{2\pi\mu^2/n}}\exp\left\{-\frac{1}{2}(\bar{x}-\mu)^2/(\mu^2/n)\right\}$$

$$\therefore \text{ if } \bar{x} = 1/n\sum_{i=1}^{n}x_i \Rightarrow f(\underline{x}\mid\bar{X}=\bar{x};\mu) = \mu\sqrt{\frac{2\pi}{n}}\left(\frac{1}{\mu\sqrt{2\pi e}}\right)^n \exp\{\cdots\}$$

obviously $f(\underline{x}\mid\bar{X}=\bar{x};\mu)$ is not independent of μ, so $\bar{X}$ is not a sufficient statistic

(b) $$f_{\underline{X}}(\underline{x};\mu) = \underbrace{\left(\frac{1}{\mu\sqrt{2\pi e}}\right)^n \exp\left\{\left[-\frac{1}{2\mu^2} \quad \frac{1}{\mu}\right]\left[\sum_{i=1}^{n}x_i^2 \quad \sum_{i=1}^{n}x_i\right]^T\right\}}_{g(T,\theta)} \times \underbrace{1}_{h(\underline{x})}$$

Sufficient statistic $\underline{T} = \left[\sum_{i=1}^{n}x_i^2 \quad \sum_{i=1}^{n}x_i\right]$

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(c) $f_T(t;\theta) = \sum_{\underline{x}} f_T(t \mid \underline{X} = \underline{x}; \theta)\, f_{\underline{X}}(\underline{x};\theta)$

$$f_T(t \mid \underline{X} = \underline{x}; \theta) = \begin{cases} 1, & \text{if } T(\underline{x}) = t \\ 0, & \text{if } T(\underline{x}) \neq t \end{cases}$$

$$\therefore\ f_T(t;\theta) = \sum_{\{\underline{x}:T(\underline{x})=t\}} f_{\underline{X}}(\underline{x};\theta) = g(t,\theta) \underbrace{\sum_{\{\underline{x}:T(\underline{x})=t\}} h(\underline{x})}_{q(t)}$$