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ECE 7251: Signal Detection and Estimation
A Hard K-L Expansion Problem

1 Problems

1. Consider the Ornstein-Uhlenbech process which has zero mean and covariance $C_x(s,t) = \exp(-\beta|t-s|)$, where $\beta > 0$, truncated to the interval $[0,1]$.

   (a) Give the integral equation which the eigenfunctions and the eigenvalues of the Karhunen-Loève expansion of X_t must satisfy. You will want to express one side as the sum of two nonoverlapping integrals.

   (b) By taking derivatives (remember the Leibnitz rule!), show that the eigenfunctions and eigenvalues of the Karhunen-Loève expansion must satisfy the differential equation

   $$\phi'' + \left(\frac{2\beta}{\lambda} - \beta^2\right)\phi = 0.$$

   (c) Looking at our differential equation, we see that we could express our eigenfunctions as $\phi(t) = a\cos(\omega t) + b\sin(\omega t)$. This makes the calculus a bit of a pain, though, so let's instead use the equivalent representation

   $$\phi(t) = c_1 e^{j\omega t} + c_2 e^{j\omega t} \tag{1}$$

   $$\omega = \sqrt{\frac{2\beta}{\lambda} - \beta^2}$$

   Substitute (1) into your expression for part (a), and perform the integrals to get a new equation.

   (d) Using your answer in part (c), show that the eigenvalues are

   $$\lambda_k = \frac{2\beta}{\omega_k^2 + \beta^2},$$

   where $\omega_1, \omega_2, \ldots$ are the positive solutions of the transcendental equation

   $$e^{2j\omega} = \left(\frac{\beta - \omega j}{\beta + \omega j}\right)^2$$

   or equivalently

   $$\cos(2\omega) = \frac{\beta^4 - 6\beta^2\omega^2 + \omega^4}{(\beta^2 + \omega^2)^2}$$

   Note that the eigenfunctions are sinusoids whose frequencies are *not* harmonically related!

   (To give credit where it's due: I took this example from pp. 67-68 of Ulf Grenander's *Abstract Inference,* which is a book that is just as intense and scary as you would expect it to be, having a title like "Abstract Inference." A similar example, except with the expansion done from $-T$ to T instead of from 0 to T, is on pp. 187-189).

2 Partial Solution

1. We have

   (a)

$$\lambda\phi(s) = \int_0^s e^{\beta(t-s)}\phi(t)dt + \int_s^1 e^{\beta(s-t)}\phi(t)dt$$

   (b) Taking derivatives once with respect to s yields:

$$\begin{aligned}\lambda\phi'(s) &= \phi(s) - \beta\int_0^s e^{\beta(t-s)}\phi(t)dt - \phi(s) + \beta\int_s^1 e^{\beta(s-t)}\phi(t)dt \\ &= -\beta\int_0^s e^{\beta(t-s)}\phi(t)dt + \beta\int_s^1 e^{\beta(s-t)}\phi(t)dt\end{aligned}$$

   Taking derivatives a second time yields:

$$\begin{aligned}\lambda\phi''(s) &= -\beta\phi(s) + \beta^2\int_0^s e^{\beta(t-s)}\phi(t)dt - \beta\phi(s) + \beta^2\int_s^1 e^{\beta(s-t)}\phi(t)dt \\ &= -2\beta\phi(s) + \beta^2(\int_0^s e^{\beta(t-s)}\phi(t)dt + \int_s^1 e^{\beta(s-t)}\phi(t)dt) \\ &= -2\beta\phi(s)\beta^2\lambda\phi(s) \\ &= \phi(s)(\lambda\beta^2 - 2\beta) \\ \phi''(s) &= \left(\beta^2 - \frac{2\beta}{\lambda}\right)\phi(s)\end{aligned}$$