COMPLEXITY REGULARIZED SHAPE ESTIMATION FROM NOISY FOURIER DATA

Natalia A. Schmid, Yoram Bresler, and Pierre Moulin *

University of Illinois at Urbana-Champaign
Coordinated Science Laboratory, 1308 West Main, Urbana, IL 61801
nschmid@ifp.uiuc.edu, ybresler@uiuc.edu, moulin@ifp.uiuc.edu

ABSTRACT

We consider the estimation of an unknown arbitrary 2D object shape from sparse noisy samples of its Fourier transform. The estimate of the closed boundary curve is parametrized by normalized Fourier descriptors (FDs). We use Rissanen's MDL criterion to regularize this ill-posed non-linear inverse problem and determine an optimum tradeoff between approximation and estimation errors by picking an optimum order for the FD parametrization. The performance of the proposed estimator is quantified in terms of the area discrepancy between the true and estimated object. Numerical results demonstrate the effectiveness of the proposed approach.

1. PROBLEM STATEMENT

Various applications, including magnetic resonance imaging, tomographic reconstruction, and synthetic aperture radar (SAR) involve estimation of an object shape from noisy sparse Fourier data. A similar Fourier formulation applies to linearizations of nonlinear inverse scattering problems using Born or physical optics approximations [1]. Depending on the amount of Fourier data available, conventional reconstruction based on Fourier inversion can lead to severe artifacts or even useless images. In this work, we propose instead a method based on statistical inference to reconstruct the shape of the object.

Suppose that an object of interest with unknown support D is located somewhere in a two-dimensional scene with a finite support $\Omega \subset \mathcal{R}^2$. The scene is described by two *known* continuous intensity functions $f(x,y)$ for $(x,y) \in D$ and $g(x,y)$ for $(x,y) \in \Omega \setminus D$, respectively. The closed boundary ∂D of the unknown shape can be represented as a vector function $\mathbf{s}(t) = [x(t), y(t)]^T$, $t \in [0,1]$, with $x(t)$ and $y(t)$ real periodic functions.

The continuous Fourier transform of the scene

$$\begin{aligned} G_s(u,v) = \int\int_{D_s} f(x,y) e^{-j2\pi(xu+yv)} dx\,dy \\ + \int\int_{\Omega\setminus D_s} g(x,y) e^{-j2\pi(xu+yv)} dx\,dy \end{aligned} \quad (1)$$

is a deterministic functional of the unknown boundary $\mathbf{s}(t)$, $t \in [0,1]$. The observations

$$Y_n = G_{\mathbf{s}}(u_n, v_n) + \epsilon_n, \;\; n = 1, \ldots N \quad (2)$$

are noisy samples of $G_s(u,v)$ at fixed and known spatial frequencies $\Delta = \{u_n, v_n, \; n = 1, \ldots N\}$, corrupted by a white complex Gaussian noise sequence ϵ_n with variance σ^2.

The problem is to estimate the unknown boundary $\mathbf{s}(t)$, $t \in [0,1]$ from the noisy observations.

2. COMPLEXITY-REGULARIZED ESTIMATOR

Shape estimation from a finite amount of data is in general an ill-posed problem. To obtain a regularized estimator, we apply a penalized-likelihood approach. A variety of known contour estimation methods used in image processing and computer vision (see [2–4] and references there in) are based on penalized-likelihood techniques. The major limitation of many of these methods is their non-adaptiveness. In this work, we use Rissanen's MDL criterion [6]. It is one of fully automated estimation principles which are broadly applied in signal, image, and contour estimation [3–5].

Since both functions composing $\mathbf{s}(t)$ are periodic on $[0,1]$, the boundary estimate can be represented using a Fourier series. The Fourier coefficients that describe a two dimensional boundary, are called Fourier descriptors (FD) (cf. [7] and references there in), and the parametrization can be used to describe arbitrarily located general-shaped objects.

Let $\mathcal{S}^{(\nu)}$, $\nu = 1, 2, 3, \ldots$, denote a set of vector functions of the form

$$\mathbf{s}^{(\nu)}(t) = \left[\begin{array}{c} a_0 + \sum_{k=1}^{d} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \\ c_0 + \sum_{k=1}^{d} c_k \cos(2\pi kt) + d_k \sin(2\pi kt) \end{array} \right], \quad (3)$$

where $b_1 = c_1$ to guarantee uniqueness of the parametrization (4) [7]. Denote by θ the combined column vector $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$, taking values in $\mathcal{R}^{4\nu+1}$, and by $\mathbf{B}(t)$ the full rank matrix of dimension $2 \times (4\nu + 2)$ containing the sin

*This work was supported by a grant from DARPA under Contract F49620-98-1-0498, administered by AFOSR.

and $\cos$ functions, so that the two dimensional vector function $\mathbf{s}^{(\nu)}(t)$ is given by $\mathbf{s}^{(\nu)}(t) = \mathbf{B}(t)\theta$, $t \in [0,1]$.

Now, for given ν, one can determine the maximum likelihood (ML) estimate of the parameters, $\hat{\theta}$, as the solution to the following maximization problem:

$$\hat{\theta} = \arg \max_{\theta \in \mathcal{R}^{4\nu+1}} \log p(\mathbf{Y}|\mathbf{s}^{(\nu)}(\theta)),\ \mathbf{s}^{(\nu)}(\theta) \in \mathcal{S}^{(\nu)}. \quad (4)$$

The remaining problem, is to determine the "correct" order ν of the FD model. In general, increasing ν , allows to capture increasing spatial detail in the represented boundary. Standard results on approximation by Fourier series can be used to provide rates of convergence with ν of the boundary approximation, for a given class of boundary functions $x(t)$ and $y(t)$. Here, we assume that the boundary is sufficiently regular that the approximation error decays with increasing order ν. However, with limited and noisy data, increasing ν beyond a certain point leads to diminishing returns, because of increase in the estimation error of more parameters from a fixed amount of data. Thus, once the FD model has been picked for the boundary, regularization reduces to selection of the model order ν.

We use Rissanen's MDL criterion to determine an optimum model order ν. The MDL criterion is stated for our problem as follows:

$$\hat{\nu}(\mathbf{Y}) = \arg \min_{\nu=1,2,\dots} \left[-\log p(\mathbf{Y}|\hat{\mathbf{s}}^{(\nu)}) + L(\hat{\mathbf{s}}^{(\nu)}) \right], \quad (5)$$

where $\hat{\mathbf{s}}^{(\nu)}(t) = \mathbf{B}(t)\hat{\theta}$ is the estimate of the boundary with $(4\nu+1)$ ML-estimated parameters substituted in (3) in place of the unknown parameters, and $L(\hat{\mathbf{s}}^{(\nu)})$ is a penalty term equal to the code length of $(4\nu+1)$ estimated parameters.

A variety of coding schemes can be considered to describe the penalty term in (5) . Under the practical condition of a large sample size, the description length of each parameter can be approximated [6] by $1/2 \log M$, with M being the number of data samples. Assuming that every model parameter is equally penalized, this leads to $L(\hat{\mathbf{s}}) \sim (4\nu+1)/2\log(2N)$, where $2N$ is the total number of real valued samples.

While the MDL criterion has well-established asymptotic optimality properties [6], it is important to evaluate the performance of the shape estimator using a geometrically meaningful criterion. In this work, we use a special discrepancy measure, the TOtal Misclassified Image Area (TOMIA) to quantify the performance of the shape estimator. TOMIA is defined as

$$\begin{aligned} TOMIA &= \left|D \cup \hat{D}\right| - \left|D \cap \hat{D}\right| \\ &= \int \int_{(x,y)\in\Omega} \left|\chi_{(x,y)\in D} - \chi_{(x,y)\in\hat{D}}\right|^2 dx\,dy, \end{aligned} \quad (6)$$

where D and $\hat{D}$ are the true and the estimated shapes and $\chi(\cdot)$ is the indicator function. If the boundaries of both shapes have parametric representations, then after invoking Parseval's equality, (6) reduces to

$$TOMIA = \int\int |H_{\mathbf{s}}(u,v) - H_{\hat{\mathbf{s}}}(u,v)|^2\, du dv, \quad (7)$$

where $H_{\mathbf{s}}(u,v)$ is the analog of $G_{\mathbf{s}}(u,v)$ in (1) with $f(x,y) = 1$ and $g(x,y) = 0$. Note that TOMIA cannot be computed without knowing the actual object and thus is not applicable for choosing the optimal model order in practical situations. In practice it is important to choose the optimal model order based on a single data observation with almost no knowledge about the actual shape.

3. IMPLEMENTATION ISSUES

For the complex Gaussian model described in (2), the optimization problem (4) can be reduced to the solution of the following non-linear least-squares problem

$$\hat{\theta} = \arg \min_{\theta \in \mathcal{R}^{4\nu+1}} \sum_{n=1}^{N} \frac{|Y_n - G_{\mathbf{s}^{(\nu)}}(u_n, v_n)|^2}{\sigma^2}, \quad (8)$$

$\nu = 1, \dots, \nu_{\max}$, where $\nu_{\max}$ is the maximal finite model order used in computations, $G_{\mathbf{s}^{(\nu)}}(u,v)$ is given by (1) with $\mathbf{s}$ replaced by $\mathbf{s}^{(\nu)} \in \mathcal{S}^{(\nu)}$.

The functional $G_s(u,v)$ in (1) depends implicitly on the boundary $\mathbf{s}$. To make this dependence explicit, we apply Green's theorem [8] and then reduce the integral to the following form:

$$\begin{aligned} G_s(u,v) = \mathcal{C}(u,v) + \tfrac{1}{2}\times \\ \Big[\int_0^1 \int_0^{x(t)} h(\alpha, y(t))\dot{y}(t) e^{-j2\pi(\alpha u + y(t)v)} d\alpha dt \\ - \int_0^1 \int_0^{y(t)} h(x(t),\alpha)\dot{x}(t) e^{-j2\pi(\alpha v + x(t)u)} d\alpha dt \Big], \end{aligned}$$

where $h(x,y) = f(x,y) - g(x,y)$, $\mathcal{C}(u,v)$ is not a function of the unknown shape, and $\dot{x}(t)$ and $\dot{y}(t)$ are the derivatives of functions $x(t)$ and $y(t)$ with respect to t.

To implement (8) with the functional $G_{\mathbf{s}^{(\nu)}}(u,v)$, $\mathbf{s}^{(\nu)} \in \mathcal{S}^{(\nu)}$, above, we invoke the function *fminunc.m* from the MATLAB optimization toolbox. This function performs unconstrained nonlinear optimization based on a quasi-Newton method. Initial experiments have shown that the algorithm exhibits instabilities: it is sensitive to the initial guess, especially for low signal-to-noise ratio (SNR), defined here as $SNR = 10\log_{10} \frac{1}{N\sigma^2} \sum_{n=1}^{N} |\ G_{\mathbf{s}}(u_n, v_n)\ |^2$. In this case, the algorithm often results in a self intersecting boundary. To avoid this problem, we propose to include an additional penalty term in (4) that penalizes rough boundaries:

$$\begin{aligned} \hat{\theta} = \arg\min_{\theta \in \mathcal{R}^{4\nu+1}} \Big[\tfrac{1}{\sigma^2} \textstyle\sum_{n=1}^{N} |Y_n - G_{\mathbf{s}}(u_n, v_n)|^2 \\ + \alpha \int_0^1 \left([\dot{x}(t)]^2 + [\dot{y}(t)]^2 \right)^{1/2} dt \Big], \end{aligned} \quad (9)$$

where α is a regularization parameter. Consider a sequence of regularization problems in the form (9) parameterized by a sequence of parameters α_l gradually diminishing to zero. Under this setting, the solution to the problem in (9) converges to the solution of the non-regularized problem in (4), as $\alpha_l \to 0$. The procedure described above is implemented recursively, that is, the solution to the optimization problem (9) with larger α is used as an initial guess to the optimization problem with a smaller value of α. This leads to a stable solution and helps avoid the occurrence of boundary self intersections.

4. NUMERICAL RESULTS

The MDL criterion is first applied to find the estimate of the airplane shape displayed in broken line in Fig. 1. This shape is described by an infinite number of FDs. We assume a binary valued intensity function: $f(x, y) = 1$ and $g(x, y) = 0$. The observed data is a 21×21 matrix of Fourier samples generated by applying the continuous Fourier transform to the image (defined as a continuous function), and uniformly sampling the transformed image $G_s(u, v)$ at intervals 0.01 in the u and in v directions. Samples of a complex white Gaussian noise of variance $\sigma^2 = 0.2$ are added to each Fourier sample resulting in SNR=29.9 dB. We performed 100 Monte-Carlo runs, each with a different noise realization. In each run, we obtained the estimates of the airplane shape, the value of the description length for all values of $1 \leq \nu \leq 8$, and to characterize the estimation performance, also calculated the value of the TOMIA.

The results of one such Monte-Carlo run are shown in Fig. 1, which depicts four reconstructed shapes parametrized by 2, 4, 6, and 8 FDs. The MDL shape estimate is in the lower left corner. The estimated shape is oversmoothed, but conveys useful information about the size, location, and shape of the airplane.

Averages of the description length and TOMIA over all 100 Monte-Carlo runs are displayed in Fig. 2 as a function of the number of FDs. Like other measures of estimation performance, the TOMIA can be decomposed into the estimation and the approximation error terms. A valley in the plot of the average TOMIA as a function of the number of FDs represents a trade-off between these error terms. The number of FDs that minimizes the average TOMIA, is the optimum model order in the average-TOMIA sense. Since in our simulations, the average TOMIA has a broad and flat valley, the values of ν producing this valley represent the approximately correct model in the average-TOMIA sense. In our simulations, in 2 out of 100 cases, the MDL criterion picked the model order $\nu = 5$, resulting in probability of underestimation of 2%. In the other 98 cases the MDL criterion picked the model order $\nu = 6$, that is, the correct model order. Thus, in this experiment, the MDL picks with high probability the least complex model out of the sequence of optimal models in the average-TOMIA sense.

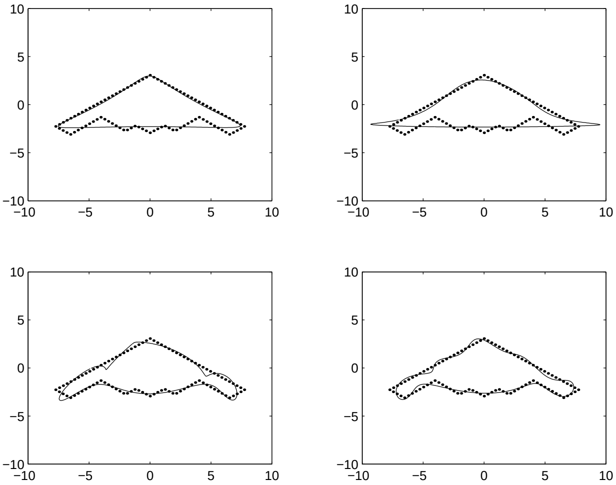
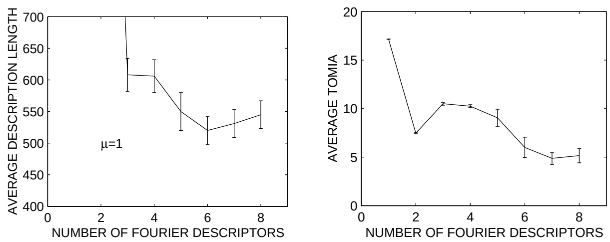


Fig. 1. Estimates of the airplane contour parameterized by $\nu = 2, 4, 6$, and 8 FDs. True contour shown in dotted line. The MDL criterion picks the shape displayed in the lower left corner ($\nu = 6$ FDs).



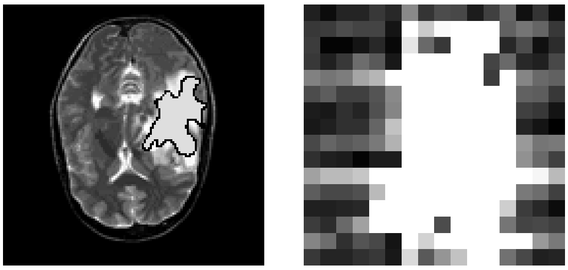
(a) (b)

Fig. 2. (a) Average description length, and (b) average TOMIA as a function of the model order ν of plane shape estimate.

For the second experiment, the shape to be estimated is a brain tumor of known grey level $f(x, y) = 1$, occluding a known background $g(x, y)$ (representing, e.g., a baseline reference scan) of the brain without a tumor (see Fig. 3-(a)). Again, the true shape is represented by a FD model of infinite order.

For this experiment we consider a single realization of noisy Fourier data, at SNR= 32 dB. The 16×16 matrix of Fourier samples was taken with intervals 0.02 in the u and v directions. Fig. 4 shows four shape estimates obtained from this data using $\nu = 3, 7, 16$, and 23 FDs. The shape displayed in the lower right is the MDL estimate ($\nu = 23$ FDs), and provides a fairly accurate depiction of the tumor. In contrast, conventional inverse Fourier reconstruction shown in Fig. 3-(b) from this limited set of Fourier data suffers from severe artifacts.

The description length and the TOMIA for this simula-



(a) (b)

Fig. 3. (a) Original discretized image of brain tumor occluding a known brain background. (b) Inverse Fourier reconstruction.

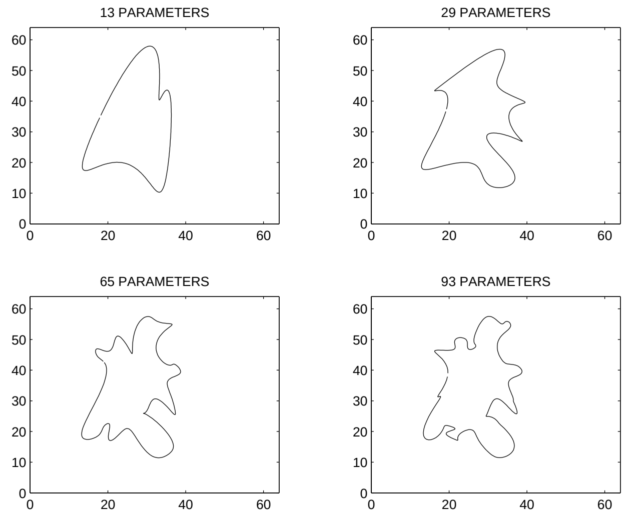
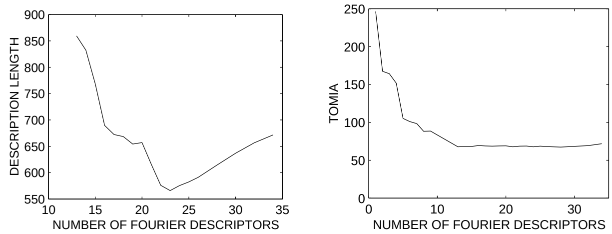


Fig. 4. Shape estimates parameterized by $\nu = 3, 7, 16$, and 23 FDs. The MDL estimate of the true shape is shown in the lower right ($\nu = 23$ FDs).

tion are displayed in Fig. 5. The minimum of the description length is achieved at $\nu = 23$ FDs and the minimum of the TOMIA is achieved at several values of ν starting from $\nu = 12$. Thus, based on the TOMIA criterion, MDL picked a correct model order.

5. CONCLUSION

Several features distinguish this work from previous work on the adaptive shape estimation:
1. We consider a linear inverse problem with nonlinear involvement of unknown parameters. The data are noisy samples of continuous Fourier transform of a scene which includes an object of interest.
2. To stabilize estimates during the implementation, we introduce an additional regularization term: a roughness penalty. To guarantee convergence of sequence of estimates to the true estimate, we gradually reduce the penalty coefficient to zero.



(a) (b)

Fig. 5. (a) description length and (b) the TOMIA as a function of the number of FDs.

3. In previous application of MDL to image processing, the performance was often judged qualitatively, by visual evaluation of the results on some test images. We use a quantitative performance measure, TOMIA, and evaluate it in systematic Monte-Carlo experiments.

6. REFERENCES

[1] F. Natterer, *The Mathematics of Computerized Tomography*, SIAM, 2001.

[2] A. Blake and M. Isard, *Active Contours*, Springer, New York, 1998.

[3] Song Chun Zhu and Alan Yuille, ”Region Competition: Unifying Snakes, Region Growing, and Bayes/MDL for Multi-band Image Segmentation,” *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 18, no. 9, pp. 884-900, 1996.

[4] M. A. T. Figueiredo, J. M. N. Leitao, and A. K. Jain, “Unsupervised Contour Representation and Estimation Using B-Splines and MDL Criterion,” *IEEE Trans. on Image Processing*, vol. 9, no. 6, pp. 1075–1087, 2000.

[5] P. Moulin and J. Liu, “Statistical Imaging and Complexity Regularization,” *IEEE Trans. on Info. Theory*, vol. 46, no. 5, pp. 1762–1777, 2000.

[6] J. Rissanen, *Stochastic Complexity in Statistical Inquiry*, World Scientific, Singapore, 1998.

[7] J. C. Ye, Y. Bresler, and P. Moulin, “Cramer-Rao Bounds for Parametric Estimation of Target Boundaries in Nonlinear Inverse Scattering Problems,” *IEEE Trans. Antennas and Propagat.*, vol. 49 No. 5, pp. 2301 -2313, May 2001.

[8] D. V. Widder, *Advanced Calculus*, Dover Publications, Inc., New York, 1989.