# An iterative deautoconvolution algorithm for nonnegative functions

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**Abstract.** This paper considers the inverse problem of recovering a nonnegative function from its autoconvolution. We propose an algorithm that solves the problem by minimizing Csiszár's *I*-divergence between the observed autoconvolution and an estimated autoconvolution. We call it a *deautoconvolution* algorithm. Various properties of the algorithm are discussed and proven. The effectiveness of the algorithm is illustrated via numerical experiments.

## 1. Introduction

This paper considers the inverse problem of estimating a function from its autoconvolution, i.e., the convolution of that function with itself. We refer to this as the *deautoconvolution* problem. Such problems sometimes arise in physics [1]. Due to its ill-posedness, most existing solutions to the deautoconvolution problem are based on various regularization methods such as Tikhonov's regularization method or the method of Lavrent'ev. Some analytical solutions to the deautoconvolution problem have been formulated based on such regularization methods [2, 3]. Also, by noting that an autoconvolution function is an example of a linear Volterra equation of the first kind, other approaches have been proposed [4, 5, 6, 7]. Gorenflo and Hofmann [2] studied theoretical aspects of the autoconvolution operator.

Most work on the subject has focused on analytical solutions. We instead present an iterative algorithm that tries to minimize an objective function. This work focuses specifically on the case where both the underlying function and its autoconvolution are nonnegative. We formulate an information-theoretic discrepancy measure between the observed autoconvolution data and the autoconvolution of the estimate of the desired function at the current iteration. Csiszár's *I*-divergence is used for defining such a discrepancy.

This discrepancy is minimized using the Kuhn-Tucker conditions. Because no closed-form solution is available, we find an iterative algorithm. This algorithm has a helpful set of properties. It naturally preserves support constraints, nonnegativity, and the total intensity of estimates. Such properties contribute to a proof of convergence of the difference of the two consecutive estimates of our algorithm.

### *1.1. Background on Csiszár's I-divergence*

Csiszár's *I*-divergence is a generalization of the Kullback-Leibler distance. The Kullback-Leibler distance has appeared in various fields: statistics [8, 9], pattern recognition [10, 11, 12], and spectral analysis [13]. Until Shore and Johnson [14, 15] justified, based on their four consistency axioms, the employment of the Kullback-Leibler distance in reconstruction problems, previous justifications had counted on intuitive arguments or the information-theoretic properties of the distance measure [16]. A limitation of the Kullback-Leibler distance is that it only defines a discrepancy measure between two functions that have the same integral. To compensate for this limitation, Csiszár [17] proposed his *I*-divergence measure and extended the work of Shore and Johnson to axiomatically justify using his *I*-divergence in reconstruction problems. Unlike the Kullback-Leibler distance, Csiszár's *I*-divergence measure can accommodate cases involving two functions that have different integrals. A notable result of Csiszár's work is that, if the functions involved are nonnegative, minimizing Csiszár's *I*-divergence measure is the only choice consistent with a set of intuitive postulates such as regularity, locality, and composition-consistency.

Csiszár's results inspired much work. Snyder *et al.* [18] apply the idea of minimizing Csiszár's *I*-divergence measure to image deblurring subject to nonnegativity constraints. They proposed an iterative algorithm that gives a sequence of estimates with a nice set of properties such as guaranteed convergence to the global minimum, the nonnegativity of every estimate in the sequence, and monotonically decreasing *I*-divergence. Additionally, they argued that deterministic deblurring problems with nonnegativity constraints can be thought of as statistical estimation problems from incomplete data based on an infinite number of observed samples, using the weak law of large numbers.

An important finding in [18] may be summarized as follows. Suppose some data can be modelled as a Poisson point process, and the intensity of that process is a linear transformation of an underlying point process whose intensity we wish to estimate. Assume that infinitely many data samples are available. Then, maximizing the expected value of the loglikelihood of the Poisson data is equivalent to minimizing the I-divergence between the measured mean value of the data and the estimated mean of the data, which is an output of a linear system as stated above. This finding may be interpreted in another way. If infinitely many data samples are available, finding a maximum-likelihood solution to the problem of estimating the mean of a Poisson point process is equivalent to estimating an input from an output from a linear system with a known kernel subject to nonnegativity constraints. Such an idea is generalized and rigorously formalized by Vardi and Lee [19]. Vardi and Lee concluded that a particular problem of maximum-likelihood estimation from incomplete Poisson data is equivalent to solving a linear inverse problem subject to nonnegativity constraints. This approach has been applied to deblurring problems in computerized tomography [20].

### *1.2. Motivation*

In some applications, the magnitude of the Fourier transform of an image can be measured, but not the phase. Recovering the Fourier phases from the Fourier magnitudes is equivalent to reconstructing a function from its autocorrelation, i.e. the correlation of that function with itself. Schulz and Snyder [21] used the idea of minimizing Csiszár's $I$-divergence measure to recover an image from its autocorrelation, and proposed the Schulz-Snyder phase retrieval algorithm. The success of their algorithm motivates applying the minimization of Csiszár's $I$-divergence measure to the deautoconvolution problem subject to nonnegativity constraints. As a consequence, the structure of this paper is strongly analogous to that of [21].

### *1.3. Organization*

This paper is organized as follows. Section 2 gives a mathematical framework for the problem of interest by means of the $I$-divergence measure. The algorithm is derived and discussed briefly in Section 3. Sections 4 and 5 describe and prove some properties of the algorithm. Numerical examples of reconstruction of two-dimensional images are demonstrated in Section 6. Section 7 concludes our work with brief remarks.

## 2. Problem statement

The algorithm described in this paper can be applied to any finite-dimensional function. We develop our theory in a two-dimensional space to retain reasonable generality while remaining concise.

We first introduce definitions to be used throughout this manuscript. Let $\{x(t) : t \in \mathbb{R}^2\}$ denote an input, and $\{y(s) : s \in \mathbb{R}^2\}$ denote an output produced by a nonlinear system described by

$$\int_{\mathcal{T}} x(s-t)x(t)dt = y(s), \tag{1}$$

where $\mathcal{T}$ represents the domain of $t$. We are interested in the case that $x$ is real-valued and nonnegative, which ensures that $y$ is so as well. We further assume that $x$ has a finite support, and hence so does $y$. The function $y$ is often called the *autoconvolution* of $x$ [2, 22, 23, 3]. For numerical implementation, the functions $x$ and $y$ are discretized as $\{x(n) : n \in \mathcal{N}\}$ and $\{y(m) : m \in \mathcal{M}\}$, respectively, where $\mathcal{N}$ represents the two-dimensional set $\{1, 2, \ldots, N\} \times \{1, 2, \ldots, M\}$, and $\mathcal{M}$ represents a set defined as

$$\mathcal{M} \stackrel{def}{=} \{m : m = n_1 - n_2, (n_1, n_2) \in \mathcal{N}^2\}. \tag{2}$$

Then, the discretized version of the autoconvolution is as follows:

$$y(m) = \begin{cases} \sum_{n \in \mathcal{N}} x(n)x(m-n), & m \in \mathcal{M} \\ 0, & m \notin \mathcal{M} \end{cases}. \tag{3}$$

Our goal is to reconstruct the input $\{x(n) : n \in \mathcal{N}\}$ from the measurements $\{y(m) : m \in \mathcal{M}\}$. To formalize our problem, we define an estimate of the system output, denoted by $\hat{y}$, as

$$\hat{y}(m) = \begin{cases} \sum_{n \in \mathcal{N}} \hat{x}(n)\hat{x}(m-n), & m \in \mathcal{M} \\ 0, & m \notin \mathcal{M} \end{cases}, \tag{4}$$

where $\{\hat{x}(n) : n \in \mathcal{N}\}$ denotes an estimate of the input $x$. Using this definition of $\hat{y}$, the problem of reconstructing $x$ can be stated as follows: provided that $y$ is measured, find an estimate of the input $\hat{x}$ such that $\hat{y}$ is as close as possible to $y$ in some sense. To define their closeness, we need to measure the discrepancy between $\hat{y}$ and $y$. Once the discrepancy measure, denoted by $I(y||\hat{y})$, is determined, our goal is to find an estimate $\hat{x}$ that minimizes the measure. Several feasible choices are available for $I(y||\hat{y})$, such as the cumulative absolute error. We are drawn to Csiszár's $I$-divergence measure [17], which is a generalization of the Kullback-Leibler distance. Csiszár's $I$-divergence is defined by

$$I(y||\hat{y}) = \sum_{m \in \mathcal{M}} \left\{ y(m) \ln \frac{y(m)}{\hat{y}(m)} + \hat{y}(m) - y(m) \right\}. \tag{5}$$

In (5), we define the following limiting quantities as

$$0 \ln \frac{0}{\alpha} \stackrel{def}{=} 0, \qquad 0 \ln \frac{0}{0} \stackrel{def}{=} 0, \qquad 0 \ln \frac{\alpha}{0} \stackrel{def}{=} \infty, \tag{6}$$

where $\alpha$ is an arbitrary positive constant. We assume that the measurement $y$ satisfies nonnegativity (i.e., $y(m) \geq 0$ for all $m \in \mathcal{M}$).

We now formulate our problem as follows: Given a measurement $y$, find an $\hat{x}_0$ such that

$$\hat{x}_0 = \arg\min_{\hat{x} \geq 0} I(y||\hat{y}), \tag{7}$$

where $\hat{x} \geq 0$ means that $\hat{x}(n) \geq 0$ for all $n \in \mathcal{N}$.

## 3. Deautoconvolution algorithm

Using fundamental calculus and the Kuhn-Tucker conditions, we obtain the necessary (but not sufficient) conditions for $\hat{x}_0$ to satisfy the condition in (7):

$$\frac{\partial I(y||\hat{y})}{\partial \hat{x}_0(n)} \begin{cases} = 0 & \hat{x}_0(n) > 0 \\ \geq 0 & \hat{x}_0(n) = 0 \end{cases}, \tag{8}$$

for all $n \in \mathcal{N}$. The first derivative of Csiszár's $I$-divergence measure can be obtained as follows:

$$\frac{\partial I(y||\hat{y})}{\partial \hat{x}(n)} = \sum_{m \in \mathcal{M}} [\hat{x}(m-n) + \hat{x}(m-n)] + \sum_{m \in \mathcal{M}} y(m) \frac{\hat{y}(m)}{y(m)} \left[ \frac{-y(m)\hat{y}'(m)}{\hat{y}(m)^2} \right]$$

$$= 2 \sum_{m\in\mathcal{M}} \hat{x}(m-n) - 2 \sum_{m\in\mathcal{M}} \hat{x}(m-n)\frac{y(m)}{\hat{y}(m)}. \tag{9}$$

Setting the right-hand side of the second equality in (9) equal to zero suggests the following iteration:

$$\hat{x}^{(k+1)}(n) = \hat{x}^{(k)}(n)\frac{1}{\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)} \sum_{m\in\mathcal{M}} \hat{x}^{(k)}(m-n)\frac{y(m)}{\hat{y}^{(k)}(m)}$$

$$= \hat{x}^{(k)}(n)\frac{1}{{y_0}^{1/2}} \sum_{m\in\mathcal{M}} \hat{x}^{(k)}(m-n)\frac{y(m)}{\hat{y}^{(k)}(m)}, \tag{10}$$

where

$$y_0 \stackrel{def}{=} \sum_{m\in\mathcal{M}} y(m) = \left[\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)\right]^2, \tag{11}$$

and $\hat{y}^{(k)}(m)$ is defined as

$$\hat{y}^{(k)}(m) = \sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n). \tag{12}$$

The relation given in (11) is proven in Property 4 below. When the algorithm is initialized, it should be satisfied that $\hat{x}^{(0)} \geq 0$ (zero divided by zero is defined as zero),

$$0 < \sum_{n\in\mathcal{N}} \hat{x}^{(0)}(n) < \infty. \tag{13}$$

The algorithm in (10) is very similar in form to the Schulz-Snyder phase retrieval algorithm [21].

## 4. Properties of deautoconvolution algorithm

It is desirable for estimation algorithms to incorporate known constraints, such as support or nonnegativity, on the possible solutions. Also, an iterative algorithm is hoped to produce a stable solution in the sense of [24]. Property 1–Property 3 explains how our deautoconvolution algorithm preserves nonnegativity and fixed support constraints and produces a stable solution. The proofs of the properties shown below are adapted from [21] and [24].

**Property 1. (Nonnegativity)**
*For $k = 1, 2, \ldots,$ it holds that $\hat{x}^{(k)} \geq 0$.*

*Proof.* Since $\hat{x}^{(0)}(n) \geq 0, \forall n \in \mathcal{N}$, $y(m) \geq 0, \forall m \in \mathcal{M}$, and $\hat{y}^{(0)}(m) \geq 0, \forall m \in \mathcal{M}$, it holds that $\hat{x}^{(1)}(n) \geq 0, \forall n \in \mathcal{N}$ by the definition in (10). By applying the same arguments for $k = 1, 2, \ldots$, it can be easily shown that $\hat{x}^{(k)} \geq 0, \forall n \in \mathcal{N}$. □

**Property 2. (Fixed Support)**
*If $\hat{x}^{(0)}(n) = 0$ for $n \in \mathcal{N}_1 \subset \mathcal{N}$, then $\hat{x}^{(k)}(n) = 0$ for $n \in \mathcal{N}_1$ and $k = 1, 2, \ldots$.*

*Proof.* This property follows from (10). □

**Property 3. (Fixed Minima)**
*Any estimate that satisfies the Kuhn-Tucker conditions in (8) for a minimizer is a fixed point of the deautoconvolution algorithm in (10).*

*Proof.* First, suppose that an estimate $\hat{x}^{(k)}$ satisfies the Kuhn-Tucker conditions for a minimizer given in (8) for a minimizer. Then, by the definition of the Kuhn-Tucker conditions, if $\hat{x}^{(k)}(n) > 0$ for some $n \in \mathcal{N}$, then for such $n$,

$$\frac{1}{{y_0}^{1/2}} \sum_m \hat{x}^{(k)}(m-n) \frac{y(m)}{\hat{y}^{(k)}(m)} = 1, \tag{14}$$

and hence $\hat{x}^{(k+1)}(n) = \hat{x}^{(k)}(n)$. If $\hat{x}^{(k)}(n) = 0$ for some $n \in \mathcal{N}$, then $\hat{x}^{(k+1)}(n) = 0$ for the $n$. Therefore, it holds that $\hat{x}^{(k+1)}(n) = \hat{x}^{(k)}(n), \forall n \in \mathcal{N}$. □

**Property 4. (Conservation of Total Intensity)**
*If (10) is initialized with $\hat{x}^{(0)}$ such that $\sum_{n\in\mathcal{N}} \hat{x}^{(0)}(n) = y_0^{1/2}$, then the following conditions are obtained for $k = 1, 2, \ldots$:*

$$\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n) = y_0^{1/2}. \tag{15}$$

*Proof.* Taking summation over $n \in \mathcal{N}$ on both sides of (10), we obtain the following equalities:

$$\begin{aligned}
\sum_{n\in\mathcal{N}} \hat{x}^{(k+1)}(n) &= \sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n) \frac{1}{\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)} \sum_{m\in\mathcal{M}} \hat{x}^{(k)}(m-n) \frac{y(m)}{\hat{y}^{(k)}(m)} \\
&= \frac{1}{\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)} \sum_{m\in\mathcal{M}} \frac{y(m)}{\hat{y}^{(k)}(m)} \sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n) \hat{x}^{(k)}(m-n) \\
&= \frac{1}{\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)} \sum_{m\in\mathcal{M}} \frac{y(m)}{\hat{y}^{(k)}(m)} \hat{y}^{(k)}(m) \\
&= \frac{y_0}{\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)}.
\end{aligned} \tag{16}$$

Therefore, if the $k^{th}$ estimate $\hat{x}^{(k)}$ satisfies

$$\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n) = y_0^{1/2}, \tag{17}$$

then it follows from (16) that

$$\sum_{n\in\mathcal{N}} \hat{x}^{(k+1)}(n) = y_0^{1/2}. \tag{18}$$

Hence, when the algorithm in (10) is initialized with an initial estimate $\hat{x}^{(0)}$ whose total intensity is $y_0^{1/2}$, then the conditions in (15) are satisfied by mathematical induction. □

**Property 5. (Monotonicity of *I*-divergence)**
*A sequence of estimates provided by (10) yields a sequence of I-divergence measure that is monotonically decreasing:* $I(y||\hat{y}^{(k+1)}) \leq I(y||\hat{y}^{(k)})$*, for* $k = 1, 2, \ldots$*.*

*Proof.* For $k = 0, 1, 2, \ldots$, the following relations can be drawn:

$$\begin{aligned} & I(y||\hat{y}^{(k)}) - I(y||\hat{y}^{(k+1)}) \\ &\quad = \sum_{m\in\mathcal{M}} \left[\hat{y}^{(k)}(m) - y(m)\right] + \sum_{m\in\mathcal{M}} y(m) \ln \frac{y(m)}{\hat{y}^{(k)}(m)} \\ &\quad\quad - \sum_{m\in\mathcal{M}} \left[\hat{y}^{(k+1)}(m) - y(m)\right] + \sum_{m\in\mathcal{M}} y(m) \ln \frac{y(m)}{\hat{y}^{(k+1)}(m)} \\ &\quad = \sum_{m\in\mathcal{M}} \left[\hat{y}^{(k)}(m) - \hat{y}^{(k+1)}(m)\right] + \sum_{m\in\mathcal{M}} y(m) \ln \frac{\hat{y}^{(k+1)}(m)}{\hat{y}^{(k)}(m)}. \end{aligned} \tag{19}$$

Note that, by Property 4, it holds that

$$\begin{aligned} \sum_{m\in\mathcal{M}} \hat{y}^{(k)}(m) &= \sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n) \\ &= \sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n) \sum_{n\in\mathcal{N}} \hat{x}^{(k)}(m-n) \\ &= y_0, \end{aligned} \tag{20}$$

and similarly, $\sum_{m\in\mathcal{M}} \hat{y}^{(k+1)}(m) = y_0$. Therefore, we obtain the following equation:

$$\sum_{m\in\mathcal{M}} \left[\hat{y}^{(k)}(m) - \hat{y}^{(k+1)}(m)\right] = 0. \tag{21}$$

Using (21) and (10), the relations in (19) can be rewritten as

$$\begin{aligned} I(y||\hat{y}^{(k)}) - I(y||\hat{y}^{(k+1)}) &= \sum_{m\in\mathcal{M}} y(m) \ln \frac{y(m)}{\hat{y}^{(k)}(m)} \\ &= \sum_{m\in\mathcal{M}} y(m) \ln \left[ \frac{\sum_{n\in\mathcal{N}} \hat{x}^{(k+1)}(n)\hat{x}^{(k+1)}(m-n)}{\hat{y}^{(k)}(m)} \right] \\ &= \sum_{m\in\mathcal{M}} y(m) \ln \sum_{n\in\mathcal{N}} \left[ \frac{\hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n)}{\hat{y}^{(k)}(m)} \right] \left[r^{(k)}(n) r^{(k)}(m-n)\right], \end{aligned} \tag{22}$$

where the last equality holds by the definition of the algorithm (10), and $r^{(k)}$ is defined by

$$r^{(k)}(n) = \frac{1}{{y_0}^{1/2}} \sum_{m\in\mathcal{M}} \hat{x}^{(k)}(m-n) \frac{y(m)}{\hat{y}^{(k)}(m)}. \tag{23}$$

Since the logarithm is a concave function, and it is true that

$$\frac{\hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n)}{\hat{y}^{(k)}(m)} \geq 0, \forall n; \quad \sum_{n\in\mathcal{N}} \frac{\hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n)}{\hat{y}^{(k)}(m)} = 1, \tag{24}$$

we can apply Jensen's inequality [25] to (22). Then, we obtain

$$
\begin{aligned}
& I(y||\hat{y}^{(k)}) - I(y||\hat{y}^{(k+1)}) \\
&\quad \geq \sum_{m\in\mathcal{M}} y(m) \sum_{n\in\mathcal{N}} \left[\frac{\hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n)}{\hat{y}^{(k)}(m)}\right] \ln\left[r^{(k)}(n)r^{(k)}(m-n)\right] \\
&\quad = \sum_{m\in\mathcal{M}} y(m) \sum_{n\in\mathcal{N}} \left[\frac{\hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n)}{\hat{y}^{(k)}(m)}\right] \ln r^{(k)}(n) \\
&\qquad + \sum_{m\in\mathcal{M}} y(m) \sum_{n\in\mathcal{N}} \left[\frac{\hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n)}{\hat{y}^{(k)}(m)}\right] \ln r^{(k)}(m-n) \\
&\quad = \sum_{m\in\mathcal{M}} y(m) \sum_{n\in\mathcal{N}} \left[\frac{\hat{x}^{(k)}(n)\hat{x}^{(k)}(m-n)}{\hat{y}^{(k)}(m)}\right] \ln r^{(k)}(n) \\
&\qquad + \sum_{m\in\mathcal{M}} y(m) \sum_{n'\in\mathcal{N}} \left[\frac{\hat{x}^{(k)}(m-n')\hat{x}^{(k)}(n')}{\hat{y}^{(k)}(m)}\right] \ln r^{(k)}(n') \\
&\quad = 2\sum_{m\in\mathcal{M}} y(m) \sum_{n\in\mathcal{N}} \left[\frac{\hat{x}^{(k)}(m-n)\hat{x}^{(k)}(n)}{\hat{y}^{(k)}(m)}\right] \ln r^{(k)}(n) \\
&\quad = 2\sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n) \left[\sum_{m\in\mathcal{M}} \frac{y(m)}{\hat{y}^{(k)}(m)}\hat{x}^{(k)}(m-n)\right] \ln r^{(k)}(n) \\
&\quad = 2y_0^{1/2} \sum_{n\in\mathcal{N}} \hat{x}^{(k)}(n) r^{(k)}(n) \ln r^{(k)}(n) \\
&\quad = 2y_0^{1/2} \sum_{n\in\mathcal{N}} \hat{x}^{(k+1)}(n) \ln\frac{\hat{x}^{(k+1)}(n)}{\hat{x}^{(k)}(n)} = 2y_0^{1/2} I(\hat{x}^{(k+1)}||\hat{x}^{(k)}) \geq 0,
\end{aligned} \tag{25}
$$

where $n' = m - n$, and $I(\hat{x}^{(k+1)}||\hat{x}^{(k)})$ denotes the $I$-divergence discrepancy between $\hat{x}^{(k+1)}$ and $\hat{x}^{(k)}$, which is nonnegative as a consequence of the strict concavity of the logarithm. Note that the rest of the terms in Csiszár's $I$-divergence cancel out because the integrals of $\hat{x}^{(k+1)}$ and $\hat{x}^{(k)}$ are the same. On the right-hand side in (25), the fifth and sixth equalities directly follow from the definition of the deautoconvolution algorithm. Therefore, it is proven that Csiszár's $I$-divergence measure is monotonically decreasing. □

**Corollary.** *It holds that*

$$I(y||\hat{y}^{(k+1)}) = I(y||\hat{y}^{(k)}) \tag{26}$$

*if and only if* $\hat{x}^{(k)}(n) = \hat{x}^{(k+1)}(n)$ *for all* $n \in \mathcal{N}$. *This implies that* $r^{(k)}(n) = 1$ *for all* $n \in \mathcal{N}$ *satisfying* $\hat{x}^{(k)}(n) > 0$ *if and only if (26) is satisfied.*

Property 2 warns us that we should not initialize a pixel value with zero unless we have *a priori* knowledge that the pixel should be zero. This provides a convenient way of incorporating support constraints. In the absense of other information, it is desirable to initialize the algorithm with a uniform, nonzero estimate, which seems to be the most general. (We note that with the Schulz-Snyder *deautocorrelation* algorithm, it is important not to initialize the algorithm a mirror symmetric initial estimate, and hence it is customary to add a small amount of randomness to the initial uniform estimate. We do not need to worry about that in our deautoconvolution case.) As a result of Property 5, it is guaranteed that $\hat{y}^{(k)}(m)$ is always positive for those $m$ such that $y(m)$ is positive, since if $\hat{y}^{(k)}(m)$ is zero for some $m$, then the algorithm will produce $I(y||\hat{y}^{(k)}) = \infty$ for those $m$. Hence, nonnegativity is naturally preserved.

## 5. Convergence of the difference of two consecutive estimates

This section establishes convergence of the difference of two consecutive estimates of the deautoconvolution algorithm to zero. However, this does not guarantee the convergence of the estimates to a limit point. We can show that the set of limit points is not empty, and these limit points satisfy the Kuhn-Tucker conditions, which means they are critical points. One of these critical points may be a local minimum or a saddle point. In all of our simulations, we have never observed convergence to a saddle point, but we have not proven that would always be the case in general; such explorations would involve looking at second derivatives of the $I$-divergence function, which we leave for future work.

Lemma 1 verifies that limit points of the sequence of estimates produced by our algorithm exist. Using this lemma, we show that a limit point of estimates must be a critical point using the preceding properties and a corollary along with the Kuhn-Tucker conditions.

When the algorithm is initialized as indicated in Property 4, the property imposes a constraint on the solution. In fact, using the property, we can reduce the feasible solution space. Let $\Lambda$ be the set of functions representing this reduced solution space, *i.e.*,

$$\Lambda = \left\{ \hat{x} : \sum_{n \in \mathcal{N}} \hat{x}(n) = y_0^{1/2}, \hat{x} \geq 0 \right\}. \tag{27}$$

In addition, let $\Lambda^*$ denote the set of limit points of the sequence $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ that are elements of $\Lambda$. The following lemmas and theorem will establish convergence of the difference of two consecutive estimates, $\hat{x}^{(k)}$ and $\hat{x}^{(k+1)}$. The proofs of the following lemmas and theorem are adapted from [24].

**Theorem 1. (Convergence of the difference of two consecutive estimates to zero in $\mathscr{L}_1$ norm)**
*The sequence of the difference of two consecutive estimates* $||\hat{x}^{(k+1)} - \hat{x}^{(k)}||_1$ *of the algorithm converges to zero in* $\mathscr{L}_1$ *norm.*

*Proof.* Because the $I$-divergence sequence generated by a sequence of estimates is monotonically decreasing (Property 5) and is bounded below by zero, there exists a limit $I^* \geq 0$ from the monotone convergence theorem such that [26, p. 104]:

$$\lim_{k \to \infty} I(y||\hat{y}^{(k)}) = I^*, \tag{28}$$

and moreover

$$\lim_{k\to\infty} \left\{ I(y||\hat{y}^{(k)}) - I(y||\hat{y}^{(k+1)}) \right\} = 0. \tag{29}$$

Combining (29) and (25), we obtain

$$\lim_{k\to\infty} I(\hat{x}^{(k+1)}||\hat{x}^{(k)}) = 0. \tag{30}$$

We note that the Kullback-Leibler distance is stronger than the norm $\mathscr{L}_1$ (see [27] or [28, p. 300]) in that

$$\sum_{n\in\mathcal{N}} \hat{x}^{(k+1)}(n) \ln \frac{\hat{x}^{(k+1)}(n)}{\hat{x}^{(k)}(n)} \geq \frac{1}{2\ln 2}||\hat{x}^{(k+1)} - \hat{x}^{(k)}||_1^2, \tag{31}$$

where $||\cdot||_1$ denotes the $\mathscr{L}_1$ norm (see [24, p. 299] for the definition of $\mathscr{L}_1$ norm). Since (30) is reached, the left-hand side of (31) goes to zero asymptotically. Therefore, we obtain convergence of the difference to zero in $\mathscr{L}_1$ norm:

$$\lim_{k\to\infty} \sum_{n\in\mathcal{N}} |\hat{x}^{(k+1)}(n) - \hat{x}^{(k)}(n)| = 0. \tag{32}$$

This proves the theorem. □

**Lemma 1. (Properties of Set of Limit Points)**

*Let $\mathcal{N}$ be a finite discrete set. The set of limit points $\Lambda^*$ is nonempty, compact, and connected.*

*Proof.* Our proof employs ideas from p. 371 of [24]. We first show that $\Lambda$ is closed and bounded, which means that $\Lambda$ is compact. Note that, by the equality and nonnegativity constraints on $\Lambda$, we have $\Lambda \subset \left[0, y_0^{1/2}\right]^{|\mathcal{N}|}$. Hence, $\Lambda$ is bounded. To show closedness of $\Lambda$, suppose it is not closed. Then, there exist $\overline{x} \in \Lambda$ and $\tilde{x} \in \Lambda$ such that $\overline{x} \in B(\tilde{x}, \epsilon)$ for an arbitrarily small $\epsilon > 0$. The open ball $B(\tilde{x}, \epsilon)$ is defined by (using notation in Moon [25])

$$B(\tilde{x}, \epsilon) = \{x \in \Lambda : ||\tilde{x} - x|| < \epsilon\}. \tag{33}$$

Now, we can select $\overline{x}$ such that

$$\overline{x}(n) = x(n) + \frac{\mathbf{1}_{\mathcal{N}}(n)\epsilon}{\sum_{n\in\mathcal{N}} \mathbf{1}_{\mathcal{N}}(n)}, \quad \forall n \in \mathcal{N}, \tag{34}$$

where $\mathbf{1}_{\mathcal{N}}(n)$ denotes a uniform function whose values are 1 for all $n \in \mathcal{N}$. Consequently, since $\overline{x}$ and $x$ are both nonnegative, we have

$$\begin{aligned}\sum_{n\in\mathcal{N}} \overline{x}(n) &= \sum_{n\in\mathcal{N}} \left[ x(n) + \frac{\mathbf{1}_{\mathcal{N}}(n)\epsilon}{\sum_{n\in\mathcal{N}} \mathbf{1}_{\mathcal{N}}(n)} \right] = \sum_{n\in\mathcal{N}} x(n) + \left[ \frac{\epsilon \sum_{n\in\mathcal{N}} \mathbf{1}_{\mathcal{N}}(n)}{\sum_{n\in\mathcal{N}} \mathbf{1}_{\mathcal{N}}(n)} \right] \\ &= \sum_{n\in\mathcal{N}} x(n) + \epsilon \neq y_0^{1/2}.\end{aligned} \tag{35}$$

This contradicts the assumption that $\overline{x} \in \Lambda$. Therefore, $\Lambda$ is closed. Since $\Lambda$ is closed and bounded, $\Lambda$ is compact by the Heine-Borel theorem [29]. Moreover, by the Bolzano-Weierstrass theorem [26], there exist a limit point because $\hat{x}^{(k)}(n) \in \Lambda$. Thus, $\Lambda^*$ is nonempty. The set of limit points is always closed. Since $\Lambda^*$ is a subset of a bounded set $\Lambda$, $\Lambda^*$ is also bounded. Therefore, $\Lambda^*$ is compact, by the Heine-Borel theorem.

We want to show that $\Lambda^*$ is connected. To do so, suppose it is disconnected. Then, there are at least two nonempty sets whose union is $\Lambda^*$ that are separated by the complement of the two sets. Since $\Lambda^*$ is closed, the complement is open. So, this can play a role of a disconnection. In addition, it is possible to choose a compact set $C$ that is a subset of the complement set since the complement is nonempty and open. However, elements of the sequence $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ alternate between the two disconnections, consisting of $\Lambda^*$, infinitely many times in $\Lambda$. This implies that the compact set $C$ is traversed by the elements of $\Lambda$ infinitely many times. However, we have from Theorem 1

$$\lim_{k\to\infty} I(\hat{x}^{(k+1)}||\hat{x}^{(k)}) = 0, \tag{36}$$

and hence

$$||\hat{x}^{(k+1)} - \hat{x}^{(k)}||_1 \to 0. \tag{37}$$

Therefore, $C$ must be visited by elements of $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ infinitely many times. However, $C$ is compact, and thus it contains at least one limit point of $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$. This contradicts the statement that $C \cap \Lambda^* \subset (\Lambda^*)^c \cap \Lambda^* = \emptyset$. Consequently, $\Lambda^*$ is connected. □

**Theorem 2. (Limit Points Satisfy the Kuhn-Tucker Conditions)**
*The limit point of $\hat{x}^{(k)}$ is a critical point i.e.,*

$$\lim_{k\to\infty} \hat{x}^{(k)} = \hat{x}^*, \tag{38}$$

*where $\hat{x}^*$ denotes a critical point. Since the I-divergence sequence is nonincreasing, this critical point cannot be a local maximum; it must be either a local minimum or a saddle point.*

*Proof.* Since the solution function space is defined on a finite domain, (32) also implies pointwise convergence. By Lemma 1, existence of limit points of $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ is guaranteed. Denote a limit point of $\{\hat{x}^{(k)}\}_{k=0}^{\infty}$ as $\hat{x}^*$:

$$\lim_{k\to\infty} \hat{x}^{(k)} = \hat{x}^*. \tag{39}$$

Next, we show that $\hat{x}^*$ is a critical point. Recall that the iteration is given by $\hat{x}^{(k+1)} = \hat{x}^{(k)}r^{(k)}$. If we take limit of the both sides of this equation, then we obtain

$$\hat{x}^* = \hat{x}^* r^*, \tag{40}$$

where $r^*$ denotes a limit point of $r^{(k)}$. Existence of $\hat{x}^*$ implies existence of $r^*$, since if $r^*$ diverges, then $\hat{x}^*$ diverges as well. Note that, if $\hat{x}^*$ is nonzero, $r^*$ must be one to guarantee consistency of (40). If $\hat{x}^*$ is zero, then we should have infinitely many $r^{(k)} \leq 1$ as $k$ goes to infinity. Note that

$$\hat{x}^{(k)}(n) = \hat{x}^{(0)}(n) \prod_{i=0}^{k} r^{(i)}(n), \quad \forall n \in \mathcal{N}. \tag{41}$$

The right-hand side of (41) would diverge when $k$ goes to infinity, unless we have infinitely many $r^{(k)} \leq 1$. Therefore, the limit of $r^{(k)}$ satisfies that $r^* \leq 1$. Consequently, we obtain

$$r^* = \begin{cases} = 1 & \hat{x}^* > 0 \\ \leq 1 & \hat{x}^* = 0 \end{cases} \tag{42}$$

Therefore, the Kuhn-Tucker conditions given in (8) are satisfied, and hence $\hat{x}^*$ is a critical point. □

We emphasize that the convergence of the difference between estimates does not guarantee the convergence of the estimates of the algorithm. The proof of convergence of the estimates remains as an important future research. Another question would involve the uniqueness (or lack thereof) of the limit point. Such a proof might follow along the lines of [24]. However, [24] involves a linear system with a fixed kernel; in our autoconvolution case, the equivalent kernel changes with each iteration. As a result, Cover's "alternating minimization" arguments would need to be extended. This will cause some complications in asserting the uniqueness of the limit point of the sequence produced by the deautoconvolution algorithm. So, a proof of uniqueness of the limit (if one is available) remains for future work.

## 6. Numerical examples

The preceding sections set up the mathematical foundation of our deautoconvolution algorithm. This section shows examples of some images reconstructed from their autoconvolution data. As mentioned before, we adhere to examples of two-dimensional images. The algorithm is implemented by the following sequence of steps:

(i) Begin with an input estimate $\hat{x}^{(0)}$ that is a valid image estimate (nonnegative and normalized according to Eq. (15)).

(ii) Convolve $\hat{x}^{(k)}$ with itself to obtain $\hat{y}^{(k)}$.

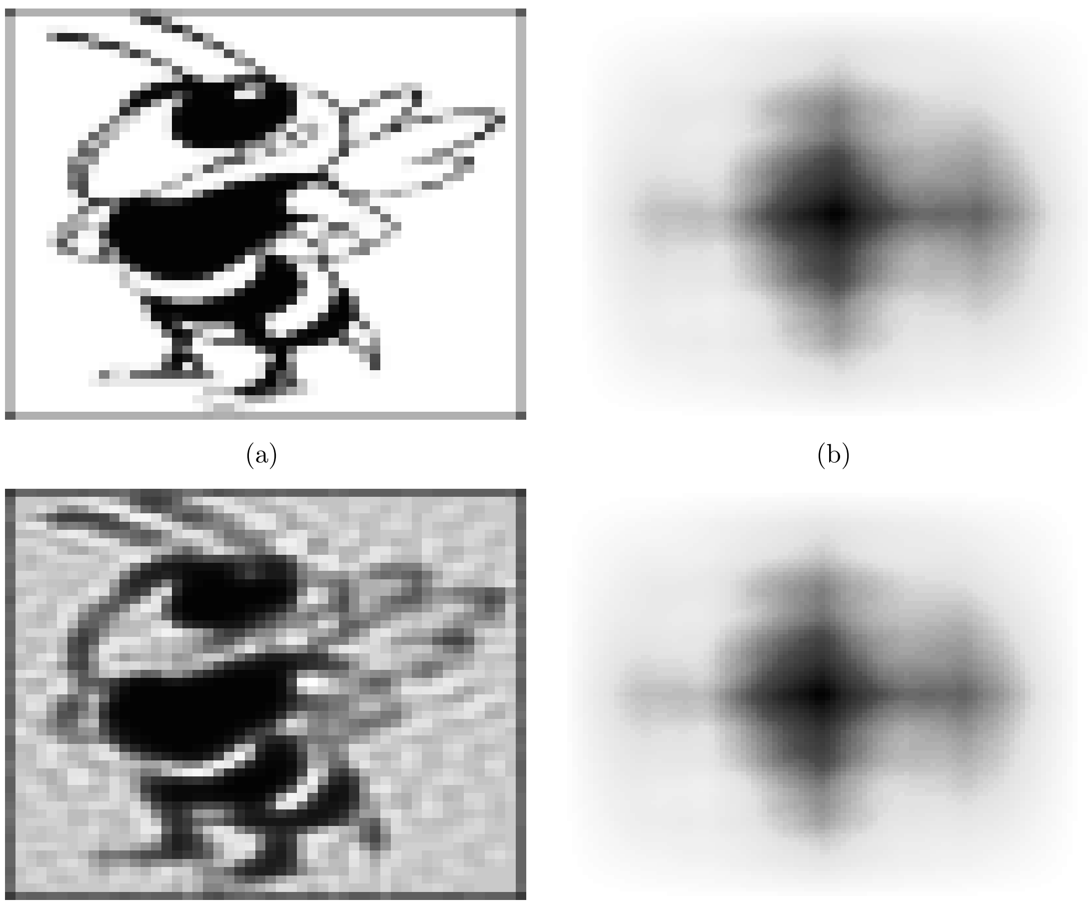
(iii) Divide the measurement $y$ by the estimated output $\hat{y}^{(k)}$. Call this function $u^{(k)}$.

(iv) Compute $r^{(k)}(n) = \frac{1}{y_0^{1/2}} \sum_{m \in \mathcal{M}} \hat{x}^{(k)}(m-n) u^{(k)}(m)$.

(v) Update the estimate of $\hat{x}^{(k)}$ by

$$\hat{x}^{(k+1)}(n) = \hat{x}^{(k)}(n) r^{(k)}(n), \ \forall n \in \mathcal{N}. \tag{43}$$

(vi) Repeat steps 2) through 4) until a convergence criterion is met.



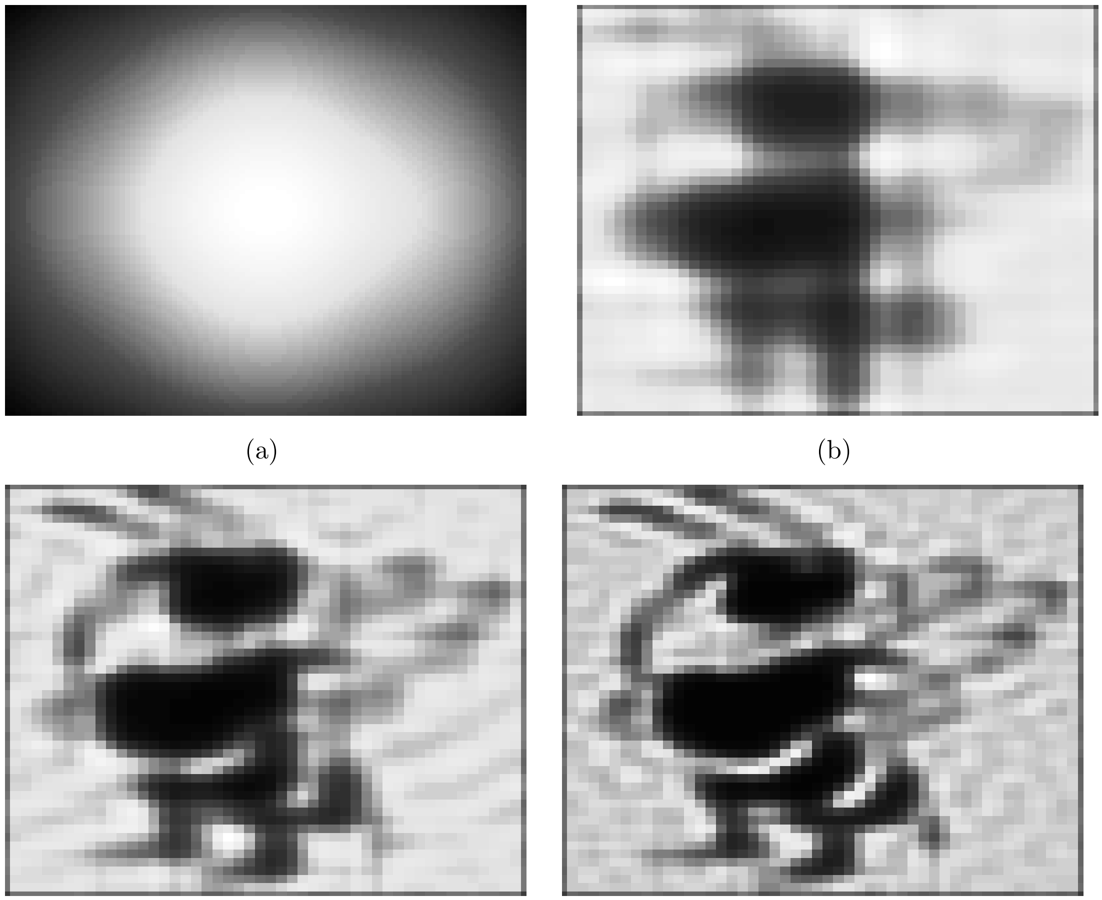
(a) (b)

(c) (d)

**Figure 1.** (a) Original image used in numerical experiments. (b) Autocorrelation of the original image. (c) Image estimate at the 20000-$th$ iteration. (d) Autocorrelation of the image estimate

Figure 1 demonstrates the reconstruction of a two-dimensional image. The size of the two-dimensional image is $50 \times 50$ pixels. Figures 1(a) and 1(c) show a two-dimensional original image and an estimate of the image provided by the algorithm at the 20000-$th$ iteration. The original image possesses various interesting details such as the feelers and wings. Figures 1(b) and 1(d) show the autoconvolutions of 1(a) and 1(c), respectively. In Figure 1, the colormaps of the yellow jacket images (we have chosen "Buzz," the mascot of the authors' institution) and the autoconvolutions are different, to best display features. For the images, black represents low values, and white represents high values; for the autocorrelations, black represents high values, and white represents low values. The estimate is remarkable in that the autoconvolution image does not show any resemblance to the original image. However, the estimate looks quite similar to the original image and shows most of details that the original image shows. The autoconvolution images look quite similar to each other as well.

Figure 2 shows some interesting, intermediate reconstructions. Our algorithm is initialized with a constant estimate, where all the values are the same and appropriately

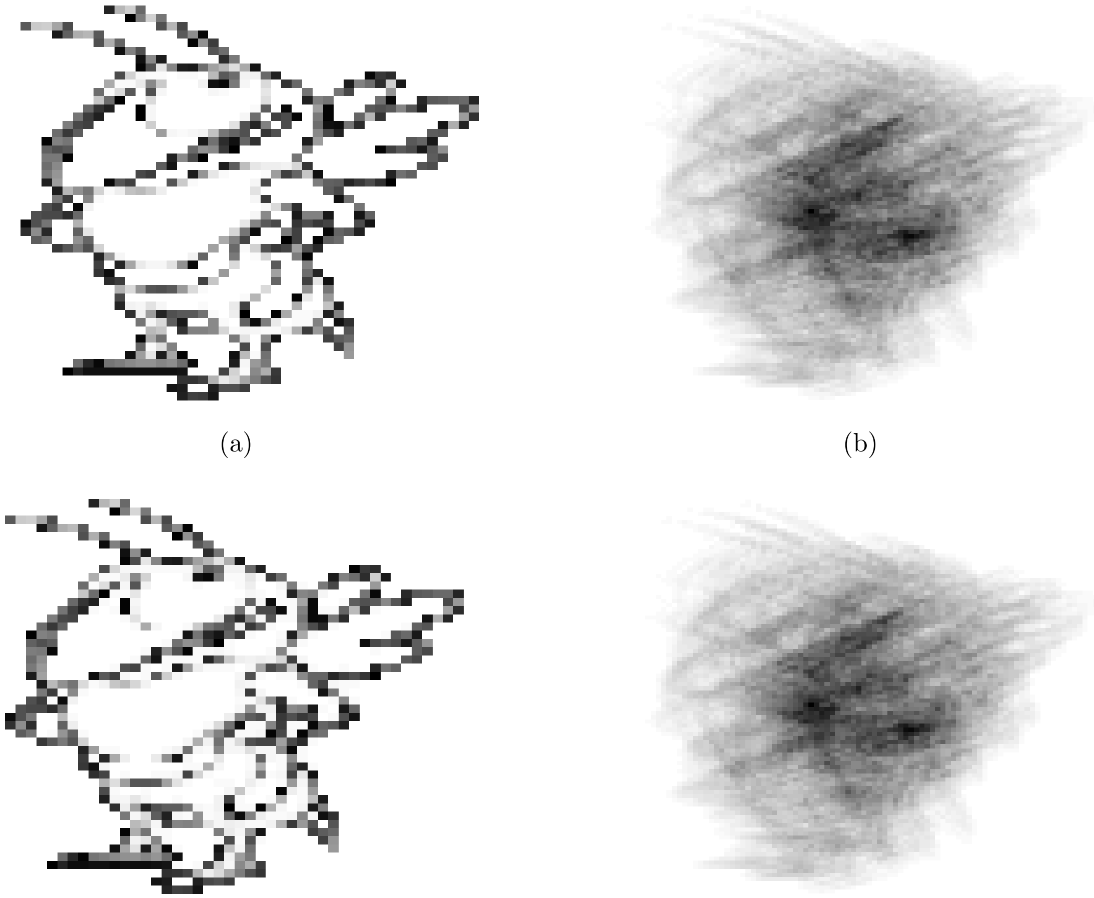


(a) (b) (c) (d)

**Figure 2.** Selected reconstructions of Figure 1(a) at the 1-*st* (a), 500-*th* (b), 5000-th (c), and 15000-th (d) iteration.

scaled. Note how different the estimates at the first iteration and at the 15000-*th* iteration are. At the 5000-*th* iteration, the yellow jacket is already somewhat identifiable. Although the solution at the 5000-*th* iteration is usable, we run the algorithm to the 20000-*th* iteration until changes in the estimate are hardly observable.

Figures 3(a) and 3(c) show a second test image, created by extracting edges from the yellow jacket, and an estimate of this image after 1000 iterations of the deautoconvolution algorithm. Notice that the estimate looks almost the same as the edge-extracted image. The autoconvolutions of them are almost the same as well. Here, in Figure 3, the colormaps for the original images and the autoconvolutions are the same (unlike in the previous example). Black represents high values, and white represents low values. Figures 3(b) and 3(d) show the autoconvolution images of Figures 3(a) and 3(c), respectively. It is interesting that the estimates of the edge-extracted image converge much faster than the original image in Figure 1(c). Noting that we started with the same initial estimate, we might conjecture that the dimension of the space of nonzero valued parameters in the image affects the speed of convergence. Figure 4 shows some selected iterations. As in Figure 2, the estimates in the earlier stage look like blurred



(a) (b) (c) (d)

**Figure 3.** (a) Original image used in numerical experiments. (b) Autocorrelation of the original image. (c) Image estimate at the 1000-*th* iteration. (d) Autocorrelation of the image estimate

versions of the estimates in the later stage. As before, the algorithm is run to the 1000-*th* iteration to obtain a "visibly best" image.

## 7. Conclusions

We have proposed a *deautoconvolution* algorithm that estimates a nonnegative function from its autoconvolution. Since our deautoconvolution algorithm has basically the same mathematical foundation as the Schulz-Snyder phase retrieval algorithm and Cover's algorithm for maximizing log-investment return of a portfolio, most of our mathematical proofs are based on their work [21, 24].

The algorithm naturally incorporates constraints on the solution such as nonnegativity and known image support. The algorithm also possesses other nice properties such as guaranteed monotonically decreasing *I*-divergence and conservation of total intensity, which implies a reduced space of solutions. Furthermore, convergence

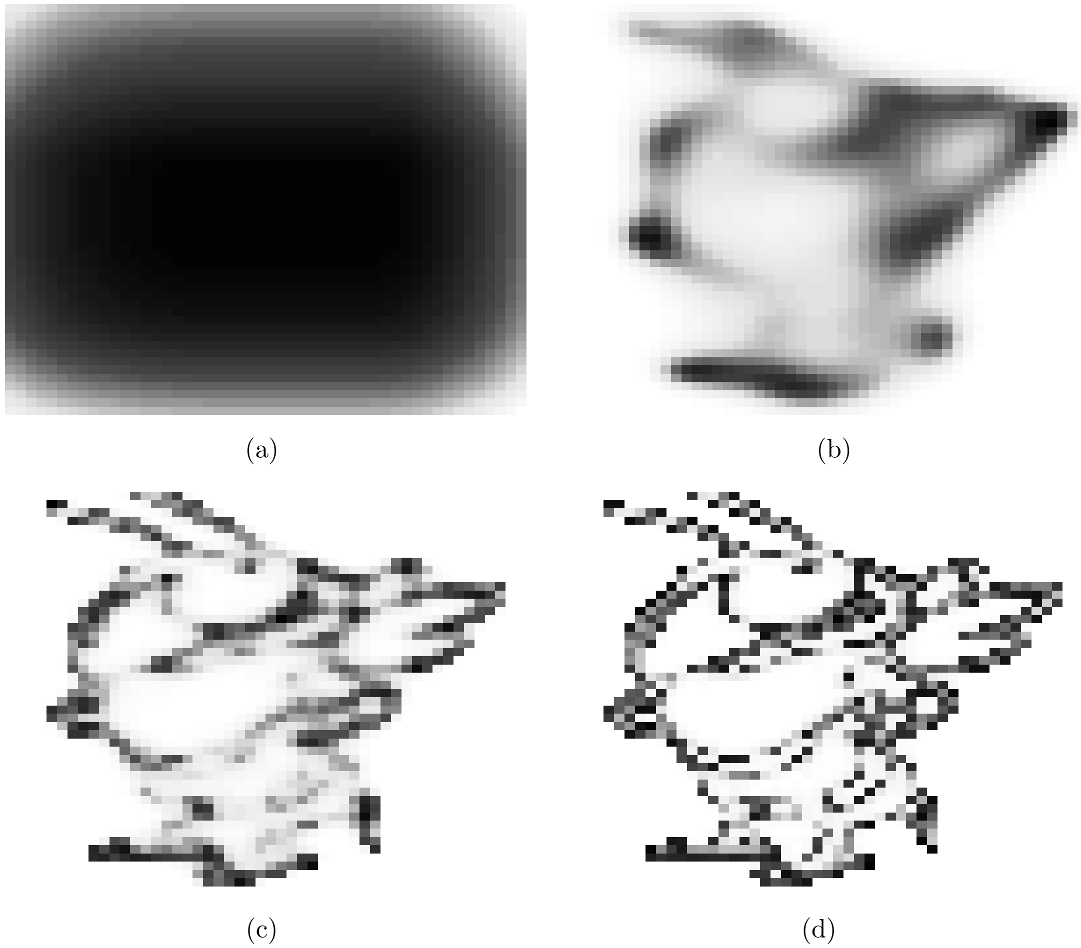


**Figure 4.** Selected reconstructions of Figure 3(a) at the 1-*st* (a), 70-*th* (b), 300-*th*, and 700-*th* iteration.

of the difference of two consecutive estimates of the algorithm to zero has been shown. Also, we have analytically shown that a limit point of the estimates of the algorithm is a critical point (either a local minimum or a saddle point). Although we might conjecture that the algorithm will not suffer from convergence to saddle points based on our experiments, a proof of such a conjecture (if it exists) remains for future discovery. Additional questions remain about the possibility of the algorithm becoming trapped in local, but not global, minima, as in the case of the Schulz-Snyder phase retrieval algorithm [30]. We do not know if such local minima of the I-divergence surface exist. We have not encountered any in our experiments, but that does not prove that they will never be there. This also requires further analysis.

Results from the numerical experiments are promising. The solutions provided by the deautoconvolution images are inspiringly close to the original images. Even though we do not show experiments where the measured data are corrupted by noise, our experience is that the algorithm is still robust to such cases. Studies with different level of noise remain an avenue for future work.

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