# Efficient implementation of an expectation-maximization algorithm for imaging diffuse radar targets

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## ABSTRACT

We explore a statistical view of radar imaging in which target reflectances are realizations of an underlying random process. For diffuse targets, this process is zero-mean complex Gaussian. The data consists of a realization of this process, observed through a linear transformation, and corrupted by additive noise. Image formation corresponds to estimating the elements of a diagonal covariance matrix. In general, maximum-likelihood estimates of these parameters cannot be computed in closed form. Snyder, O'Sullivan, and Miller proposed an expectation-maximization algorithm for computing these estimates iteratively. Straightforward implementations of the algorithm involve multiplication and inversion operations on extremely large matrices, which makes them computationally prohibitive. We present an implementation which exploits Strassen's recursive strategy for matrix multiplication and inversion, which may make the algorithm feasible for image sizes of interest in high-resolution radar applications.

**Keywords: statistical radar imaging, diffuse targets, maximum-likelihood estimation, expectation-maximization algorithms**

## 1. INTRODUCTION

A radar signal with sufficiently wide bandwidth may be used to develop images of the detailed structured of targets such as aircraft.[1,2] Some systems form two-dimensional images, where one dimension corresponds to range (time delay), and the other corresponds to velocity (doppler shift). The most common kind of delay-doppler radar image is formed by correlating the returned signal with time-shifted and doppler-shifted copies of the transmitted signal, with each pixel in the image corresponding to a different time-shift and doppler-shift. This is essentially a kind of matched filtering.[3] The result of this processing is known as a "cross-ambiguity function." It is an estimate of the target's reflectance as a function of delay and doppler. The magnitude or squared magnitude of this function is usually displayed as an image.

If the image consists of a signal point, this kind of matched-filter processing can be shown to locate that point in a way that is optimal from a statistical maximum-likelihood viewpoint. However, as Jakobsson, Swindlehurst, and Stoica[4] note, "Classical approaches to time-delay and Doppler estimation are based on matched filtering... Matched filtering techniques are known to be optimal in the maximum likelihood (ML) sense for a single signal arrival but are not consistent when multiple overlapping copies of the signal are present." While matched filtering suppresses noise, it also blurs the image by a system point-spread response which is the ambiguity function of the transmitted waveform, making it difficult to distinguish individual scatterers which are closely spaced. In addition, since the reflectivity is complex-valued, scatterers may interfere with each other in a destructive fashion.

To achieve higher resolution images, nonlinear technique must be invoked. A wide variety of these have been proposed. Many of these make the assumption that the target consists of a small set of discrete scattering centers. We review some of these in Sec. 1.1.1.

Here, we consider a high-resolution imaging technique proposed by Snyder, O'Sullivan, and Miller,[5] which differs in that the reflectance itself is considered a realization of some underlying random process; we seek maximum-likelihood estimates of the parameters of that underlying process. No prior assumptions or constraints are made on the allowed signal processing structure. In particular, the reflectance image is modeled as a zero-mean Gaussian random process which is independent from pixel to pixel. The variances of the pixels form the *scattering function*, which is our prime object of inference. Maximum-likelihood estimates of the scattering function are computed by an expectation-maximization (EM) algorithm.[6] Estimates of the more traditional reflectance image may be computed as a byproduct of our inference procedure.

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In the Snyder-O'Sullivan-Miller framework, no assumptions need to be made about the target consisting of a few point scatterers. Another advantage of this approach is that no assumptions need to be made about the structure of the transmitted waveform. This contrasts with most linear imaging systems, in which a stepped-frequency waveform is employed to make the matched-filter processing resemble a computationally efficient two-dimensional fast Fourier transform.

Although the statistical approach to radar imaging, and the associated EM algorithm, was developed at Washington University during the late 80's,[5] almost nothing seems to have been published on it in the last decade, with the exception of Moulin's work on regularization techniques in the special case where the mapping from the image domain to the data domain is directly invertible.[7–10] When this mapping is invertible, the algorithm greatly simplifies; when it is not, the algorithm is computationally demanding. This computational complexity is likely the prime reason the algorithm has remained dormant for so long.

Contemplating the vast strides in computer power that have been made over the past ten years, it seemed appropriate to resurrect the algorithm and try it out on modern hardware. Last year, we showed 20 by 20 radar images computed on a Sun Enterprise 3500 server using the full EM algorithm.[11] However, since a straightforward implementation of the algorithm has a computational complexity of $O(N^3)$, where $N$ is the number of pixels in the image, it seems that the usual off-hand invocation of Moore's Law will be inadequate to make the algorithm practical for large images. Hence, we expand on some ideas sketched by O'Sullivan *et al.*[12] which allow us to craft substantially more efficient implementations.

After concluding this introductory section with some discussion of related work to provide context, we formulate the radar imaging problem from a statistical viewpoint in Sec. 2. The EM algorithm for radar imaging undergoes review in Sec. 3. Section 4 details the fast matrix computation techniques we implement in order to make the algorithm practical. Section 5 displays some example images estimates using our prototype MATLAB implementation which incorporates these fast matrix procedures. Directions for future work, particularly natural ways to improve our algorithm, are proffered in Sec. 6

## 1.1. Related Work

### 1.1.1. Relevant work on reflectance estimation

When images are well-modeled as point targets, the CLEAN algorithm[13,14] is frequently used. The algorithm operates iteratively by finding the greatest peak on the image (here, the delay-doppler cross-ambiguity surface) and subtracting out the point spread response of the system (in this case, the ambiguity surface of the transmitted signal) shifted to that peak and multiplied by a user-supplied parameter.* The process then moves on to the highest remaining peak, and proceeds until a threshold has been reached. The procedure tends to work well in practice and has been thoroughly analyzed, although it is somewhat ad-hoc.

Jakobsson *et al.*[4] present a more sophisticated subspace-based approach which produces continuous parameter estimates using generalizations of the MUSIC and ESPRIT algorithms; hence, the estimates are not restricted to lie on a quantized parameter grid. Although they have communication systems in multipath environments in mind, their tools might be applicable to reflectance image formation. Their approach decouples the optimization into a two-step procedure which only requires an $N$-dimensional optimization, where $N$ is the number of points in the image. They compare the performance of their algorithm to that predicted by the Cramer-Rao bound. In a recent conference publication, Jakobsson and Swindlehurst[15] present an improved version of their algorithm which uses a more accurate time-domain model.

If a stepped-frequency waveform is employed, then the data has a two-dimensional Fourier structure. Capon's method of spectrum analysis is designed to achieve higher resolution than the FFT. While the one-dimensional Capon method is thirty years old,[16] it was successfully extended to two-dimensions only a few years ago by Liu[17] and colleagues. More recently, Jakobsson, Marple, and Stoica[18] presented a two-dimensional Capon procedure which exploits the properties of block-Toeplitz and Toeplitz-block-Toeplitz matrices.

See Ref. 19 for a broad review of the application of modern spectral analysis techniques to radar data, including techniques such as autoregressive modeling.

Maximum-entropy techniques, many of which were originated in the radio astronomy community (as was also the case with the CLEAN algorithm), may also be applied to radar imaging.[20,21] Although this is usually framed as

*Picking this parameter is something of an art, and the results are largely dependent on the skill of the practitioner.

maximizing the entropy of the magnitude image subject to fidelity to the data, it can also be viewed as a penalized least-squares problem with a pixelwise penalty.[22] Other penalty functions besides an entropy functional could be used, including ones which induce a coupling between pixels; for instance, Cetin and co-workers[23,24] employ a half-quadratic regularization scheme which smoothes over uniform regions while preserving object boundaries.

### 1.1.2. Related work on structured covariance estimation

Li *et al.* consider an asymptotic maximum likelihood (AML) approach which exploits an "extended invariance principle." In their approach, the $M$ by $M$ empirical covariance matrix of the data and the $M$ by $M$ structured covariance matrix to be estimated are both rewritten as long column vectors, each of length $M^2$. The parameters are estimated using a weighted least-squares criterion using an empirical estimate of the covariance matrix of these long vectors. This "metacovariance" matrix, which a covariance matrix of entries in a smaller covariance matrix, is $M^2$ by $M^2$. Although the size of this matrix is practical for the array processing applications considered by the authors, it is rather unwieldy for radar imaging. For instance, a data vector of length 400, as might be appropriate to form a 20 by 20 image, would involve a $400^2$ covariance matrix and hence a $400^4$ by $400^4$ metacovariance matrix with $6.56 \times 10^6$ elements. In addition, several independent data "snapshots" are needed to get a good estimate of the metacovariance; in our radar imaging scenario, we only have one snapshot. Although their method may not be directly applicable, we mention it because their theoretical results could potentially yield insight into the nature of the radar imaging problem.

## 2. PROBLEM FORMULATION

We model the received data as $\mathbf{r} = \mathbf{\Gamma}^\dagger \mathbf{c} + \mathbf{w}$, where $\mathbf{c}$ is the complex target reflectance, $\mathbf{\Gamma}$ represents a linear observation mechanism, and the receiver noise vector $\mathbf{w}$ is i.i.d. complex Gaussian with variance $N_0$. We will let $\mathbf{\Gamma}$ be rather general. For simple range profiling, the rows of $\mathbf{\Gamma}$ contain samples of the transmitted waveform; for delay-doppler imaging, it will contain the transmitted waveform multiplied by complex sinusoids associated with doppler shifts; for tomographic imaging, $\mathbf{\Gamma}$ may also represent scene rotations. In addition to radar imaging, power spectrum estimation can also be formulated this way, in which case $\mathbf{\Gamma}$ holds complex exponentials. Our algorithm will result from performing optimal statistical inference using this model, and not presuppose any particular kind of processing structure such as matched filtering.

Viewing the imaging process from a statistical angle, we may consider $\mathbf{c}$ itself to be a random variable. Different scattering mechanisms will give rise to different distributions on $\mathbf{c}$. If a sufficiently large number of scatterers give rise to a reflectance of a particular "pixel" on the target, the target is said to be "diffuse" and the reflectance is "speckled." Diffuse returns may be modeled as $\mathbf{c}_d \sim CN(0, \mathbf{\Sigma})$, where $\mathbf{\Sigma}$ is a diagonal covariance matrix. The vector of its diagonal entries $\boldsymbol{\sigma}^2 = diag(\mathbf{\Sigma})$ is called the *scattering function.*[†]

Under a purely diffuse model, $\mathbf{r} \sim CN(0, \mathbf{K_r})$ where $\mathbf{K_r} = \mathbf{\Gamma}^\dagger \mathbf{\Sigma} \mathbf{\Gamma} + N_0 \mathbf{I}$ and estimating the scattering function amounts to structured covariance estimation.[25] The loglikelihood for the data is

$$L_{id}(\mathbf{\Sigma}) = -\ln \det \mathbf{K_r} - \mathbf{r}^H \mathbf{K_r}^{-1} \mathbf{r}. \tag{1}$$

If $\mathbf{\Gamma}$ is unitary (for instance, in the case of properly chosen stepped-frequency waveforms), then one can orthogonalize the data by computing $\tilde{\mathbf{r}} = \mathbf{\Gamma r}$, and the maximum-likelihood estimate is just $\sigma_i^2 = \max(0, |\tilde{r}_i|^2 - N_0)$. If $\mathbf{\Gamma}$ is not unitary, there is no closed-form solution for the maximizer of (1). Section 3 reviews an iterative expectation-maximization algorithm for computing ML estimates of $\boldsymbol{\sigma}^2$ from $\mathbf{r}$ proposed by Snyder *et al.*[5]

### 2.1. Specialization to Delay-Doppler Imaging

Let $s(t)$ denote the continuous-time transmitted signal. We adopt the discretization technique employed by Moulin (Sec. 2.3, Ref. 7). Suppose we want to compute the delay-doppler scattering function on a discrete grid at delay times $\tau_\ell, \ell = 1, \ldots, I_R$, with a spacing of $\delta_\tau$, and at doppler frequencies $f_k, k = 1, \ldots, I_{CR}$, with a spacing of $\delta_f$. The vector $\boldsymbol{\sigma}^2$ represents the scattering function at delay $\tau_\ell$ and doppler $f_k$ with the element $\sigma^2_{f+\ell I_{CR}}$. One can imagine the $I_R$ by $I_{CR}$ two-dimensional image being unwrapped and strung along a length $I_R I_{CR}$ one-dimensional vector.

[†]In a paper presented in this same conference last year,[11] we used $\boldsymbol{\sigma}$ to represent variance in an attempt to avoid some cumbersome notation. Here, we adopt Richard Blahut's suggestion to us that we stay consistent with the engineering convention of using $\sigma^2$ to denote variance.

Suppose that $N$ samples of the radar return are taken at a spacing of $\delta_t$. The $(n, k+\ell I_{CR})^{th}$ entry of the $N \times I$ matrix $\mathbf{\Gamma}^H$ (where $I = I_R I_{CR}$) is given by (Eq. 16 of Ref. 7)

$$\exp[j2\pi f_k(n\delta_t - \tau_\ell/2)]s(n\delta_t - \tau_\ell). \tag{2}$$

## 3. AN EM ALGORITHM FOR MAXIMUM-LIKELIHOOD IMAGING

Maximum-likelihood imaging requires us to find the $\mathbf{\Sigma} = diag(\boldsymbol{\sigma})$ which maximizes (1). Since no simple solution is evident for this difficult optimization problem, we turn to an expectation-maximization algorithm derived by Snyder, O'Sullivan, and Miller[5] (see also Sec. IX.C of Ref. 26 and Sec. 3 of Ref. 11). The algorithm consists of iterating

$$\sigma_i^{2(new)} = \sigma_i^{2(old)} + (\sigma_i^{2(old)})^2[\mathbf{\Gamma}\mathbf{K_r}^{-1}\mathbf{S}\mathbf{K_r}^{-1}\mathbf{\Gamma}^H - \mathbf{\Gamma}\mathbf{K_r}^{-1}\mathbf{\Gamma}^H]_{ii}, \tag{3}$$

where $\mathbf{K_r} = \mathbf{\Gamma}^H\mathbf{\Sigma}^{old}\mathbf{\Gamma} + N_0\mathbf{I}$ and $\mathbf{S} = \mathbf{r}\mathbf{r}^H$.

Notice that the data only enters into the inference via its empirical covariance $\mathbf{S}$. We have written (3) in a way to show the symmetric structure of the matrix computations. In implementation, it is more efficient to compute the term in brackets by calculating $\Xi = \mathbf{\Gamma}\mathbf{K_r}^{old}$ followed by $\Xi(\mathbf{S}\Xi^H - \mathbf{\Gamma}^H)$. Since only the diagonal terms are needed, the final matrix multiplication only requires $I$ inner products.

An interesting byproduct of the derivation of the EM algorithm is that we can compute a conditional mean estimate of the reflectance $\mathbf{c}$ at each iteration via:

$$c_i^{(new)} = \sigma_i^{2(new)}[(\mathbf{\Gamma}^H\mathbf{\Sigma}^{new}\mathbf{\Gamma} + N_0\mathbf{I})^{-1}\mathbf{r}]_{ii} \tag{4}$$

Both the scattering function $\boldsymbol{\sigma}^2$ and the reflectance image $\mathbf{c}$ are useful representations of the target. The fact that they can both be produced simultaneously is a convenient feature of the algorithm.

## 4. FAST MATRIX COMPUTATIONS

O'Sullivan *et al.*[12] proposed implementing (3) using Strassen's algorithms[27] for matrix multiplication and inversion, which bring the complexity of these operations down from $O(N^3)$ to $O(N^{2.807})$. Here, we review Strassen's inversion method and an alternative to Strassen's original multiplication recipe developed by Winograd. We also consider ways to save computations by exploiting the Hermitian symmetry of $\mathbf{K_r}$.

### 4.1. Multilevel Matrix Multiplication

Ordinary multiplication of a 2 by 2 matrix requires eight multiplications and seven additions. Strassen[27] developed a method of multiplying 2 by 2 matrices with only seven multiplications, but 18 additions. A couple years later, Winograd[28] published a similar, slightly improved technique which uses seven multiplications and 15 additions. At first glance, it would appear that the cost of the additional number of additions could not outweigh the savings of one multiplication. However, the algorithm is perfectly applicable if the elements of the matrices are themselves submatrices; hence, the algorithm may be applied recursively. The resulting recursive algorithm has order $O(N^{\log_2 7}) = O(N^{\log 2.807})$ instead of $O(N^3)$. The constant associated with this recursive multiplication scheme, however, is large, so $N$ must be of sufficient size in order for it to have an advantage over traditional straightforward matrix multiplication. The size of matrices in the EM algorithm for diffuse-target radar imaging are well above this threshold.

The Winograd algorithm computes the matrix multiplication

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{5}$$

via the following steps:

$$
\begin{aligned}
S_1 &= A_{21} + A_{22} && (6)\\
S_2 &= S_1 - A_{11} && (7)\\
S_3 &= A_{11} - A_{21} && (8)\\
S_4 &= A_{12} - S_2 && (9)\\
T_1 &= B_{12} - B_{11} && (10)\\
T_2 &= B_{22} - T_1 && (11)\\
T_3 &= B_{22} - B_{12} && (12)\\
T_4 &= B_{21} - T_2 && (13)
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= A_{11}B_{11} && (14)\\
P_2 &= A_{12}B_{21} && (15)\\
P_3 &= S_1T_1 && (16)\\
P_4 &= S_2T_2 && (17)\\
P_5 &= S_3T_3 && (18)\\
P_6 &= S_4B_{22} && (19)\\
P_7 &= A_{22}T_4 && (20)
\end{aligned}
$$

$$
\begin{aligned}
C_{11} &= P_1 + P_2 && (21)\\
U_2 &= P_1 + P_4 && (22)\\
U_3 &= U_2 + P_5 && (23)\\
C_{21} &= U_3 + P_7 && (24)\\
C_{22} &= U_3 + P_3 && (25)\\
U_6 &= U_2 + P_3 && (26)\\
C_{12} &= U_6 + P_6. && (27)
\end{aligned}
$$

Due to the extra bookkeeping involved in this kind of algorithm, in implementation, one must specify a "breakpoint." Matrices smaller than this breakpoint are multiplied in the usual way; this provides the base case for the recursion. Through experimentation, we have found $N = 256$ to be a good breakpoint for our MATLAB implementation. MATLAB has a large amount of overhead for recursive calls, so a specialized implementation in a lower-level language like C would probably benefit from a lower breakpoint.

#### 4.1.1. Multiplications resulting in symmetric matrices

If the result of the multiplication is known to be symmetric, one can save a small amount of computation by calculating $C_{21}$ according to (24) and simply taking $C_{12} = C'_{21}$. Computing $U_6$, $P_6$, and $S_4$ is then unnecessary. Unfortunately, this savings will only be available at the top level of the recursion.

#### 4.1.2. Work on "faster" matrix multiplication algorithms

Although it is known that the computational complexity of matrix multiplication must be at least $O(N^2)$, the actual lower bound is unknown at this time. Nearly a decade after Strassen and Winograd's original work, Pan discovered an $O(N^{2.781})$ algorithm exploiting a technique called "trilinear aggregating." A series of incremental improvements were made in the early 80's which led to $O(N^{2.496})$ algorithms; see Ref. 29 for a comprehensive history. In 1990, Winograd and Coppersmith[30] published another algorithm which operates in $O(N^{2.37})$ time. This is the best known order. Unfortunately, these asymptotically faster algorithms are so difficult to understand, so complicated to implement, and the constants associated with their runtime so large, that they seem to be of mostly theoretical interest at present.

### 4.2. Multilevel Matrix Inversion

Strassen[27] also presented a recursive method for inverting a matrix $A$:

$$\begin{aligned} R_1 &= A_{11}^{-1} && (28)\\ R_2 &= A_{21}R_1 && (29)\\ R_3 &= R_1A_{12} && (30)\\ R_4 &= A_{21}R_3 && (31)\\ R_5 &= R_4 - A_{22} && (32)\\ R_6 &= R_5^{-1} && (33)\\ C_{12} &= R_3R_6 && (34)\\ C_{21} &= R_6R_2 && (35)\\ R_7 &= R_3C_{21} && (36)\\ C_{11} &= R_1 - R_7 && (37)\\ C_{22} &= -R_6 && (38) \end{aligned}$$

$$A^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \quad (39)$$

As with the recursive multiplication, we must specify a breakpoint for which matrices smaller than that breakpoint are computed using a more traditional matrix inversion routine. In our MATLAB implementation, we use a breakpoint of $N = 256$. As is the case with the multiplication, a more customized implementation in a lower-level language would likely benefit from a lower breakpoint.

#### 4.2.1. Recursive Inversion of Symmetric Matrices

If $A$ is Hermitian symmetric, we can avoid explicitly computing one of the diagonal entries of (39). Computing $C_{21}$ first and then taking $C_{12} = C_{21}'$ allows us to save one multiplication. However, we can save one more multiplication (29) by instead computing computing $C_{12}$ explicitly and then taking $C_{21} = C_{12}'$, since $R_2$ is not employed in any calculations other than the computation of $C_{21}$ (35). In addition, observe that the calculations of

$$R_4 = A_{21}R_3 = A_{21}R_1A_{12} = A_{21}A_{11}^{-1}A_{12} \quad (40)$$

and

$$R_7 = R_3C_21 = R_1A_{12}C_21 = R_1A_{12}C_{12}' = A_{11}^{-1}A_{12}R_6'R_3' = A_{11}^{-1}A_{12}(R_5^{-1})'A_{12}'R_1' = A_{11}^{-1}A_{12}(R_4 - A_{22})^{-1}A_{21}A_{11}^{-1} \quad (41)$$

both result in Hermitian symmetric matrices. Hence, the multiplications $R_4 = A_{21}R_3$ and $R_7 = R_3C_{21}$ benefit from the simple savings described in Sec. 4.1.1.

Fortunately, the inversions needed (Eqs. 28 and 33) themselves operate on Hermitian symmetric matrices ($A_{11}$ and $R_5 = A_{21}A_{11}'A_{12} - A_{22}$), so these savings may be obtained at each level of the recursion.

#### 4.2.2. Shulz-Jones-Mayer Inversion

For completeness, we mention the Sculz-Jones-Meyer[‡] (SJM) inversion method, which is a Newton-like technique which computes an approximate inverse $A^{-1}$. Starting with an $X$ which approximates this inverse, one iterates

[‡]We discovered this technique on a website maintained by Perfectly Scientific, Inc., at http://www.perfsci.com/fastalg/matrix/readme. Quoting from their page: "What we call the Shulz-Newton-iteration notion is referenced in Stoer & Bulirsch.[31] A. Jones and R. Mayer described their method to R. Crandall (priv. comm. 1995) for regularization of the Shulz iteration; hence our title SJM for the method." The SJM name seems to pop up in other places, such as Ref. 32, so we adopt it here.

$$X_{new} = 2X_{old} - X_{old}AX_{old}$$

The initial $X$ must be sufficiently close to $A^{-1}$ in order to avoid divergence. In the EM iterations, the parameter estimates tend to change slowly, so the inverse covariance will tend to change slowly from iteration to iteration as well. Hence, the SJM method may be helpful, because the inverse covariance from the previous iteration may be used to initialize the SJM method at the next iteration. We have not yet experimented with the SJM procedure and mention it as a potential avenue for future exploration.

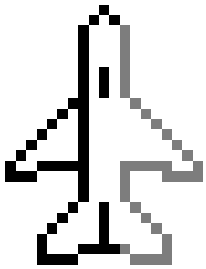

**Figure 1.** Aircraft image used in the simulations.

## 5. EXAMPLES

Figure 1 shows a cartoon representation of an airplane. This 32 by 32 image is meant solely to illustrate the behavior of the algorithm, and is not intended to resemble a real radar image of a real aircraft. The horizontal coordinate indicates range (delay), and the vertical coordinate indicates velocity (doppler); hence, this image might correspond to the aircraft rotating about its center. The higher intensity on the left is meant to indicate that side is presented to the radar; hence, the other side is shadowed.

In our simulations, we assumed a pseudorandom waveform[33] consisting of 64 segments as the transmitted signal. Each segment is independently chosen to be 1 or $-1$ with equal probability. Such waveforms are known to have good ambiguity properties (see Sec. 10.4.2 of Ref.[3]).

The upper panels of Fig. 2 show the magnitude of the data sequences produced by two specular reflectance realizations using Fig. 1 as the magnitude, with random uniformly distributed, independent phases. The application of the diffuse imaging algorithm to specular data can be justified using central limit theorem arguments; the data is "almost Gaussian" in the sense of Mallows.[34]

The lower panels of Fig. 2 shows the conventional matched filter estimates[§] $\hat{\sigma}^{\mathbf{2}} = |\mathbf{\Gamma r}|^2$ computed from the data in the upper panels. Note the blurring by the system ambiguity function. In incoherent optical imaging, a blurred object is usually still recognizable as that object; here, we would be hard pressed to see an airplane. The coherent nature the radar implies that different scatterers may interfere with each other in both constructive and destructive ways; this interference has wrecked havoc on the simple reconstruction.

Figure 3 shows the results of the first four EM iterations processing the data associated with the left column of Fig. 2, using the matched-filter estimate to initialize the algorithm. The increasing iterations are shown going from left to right. The top row shows the estimates of the scattering function $\boldsymbol{\sigma}^2$; the bottom row shows the estimates of the reflectance $\mathbf{c}$. In all of the panels, a square-root grey scale is used to help expose low-level details. Figure 4 shows the same information for the data associated with the right column of Fig. 2. Notice that a sharp image of the plane appears after surprisingly few iterations. The effect of increasing iterations is particularly intriguing. Concerning the scattering function, the increasing iterations seem to reduce smeared, blurry masses in the background. For the

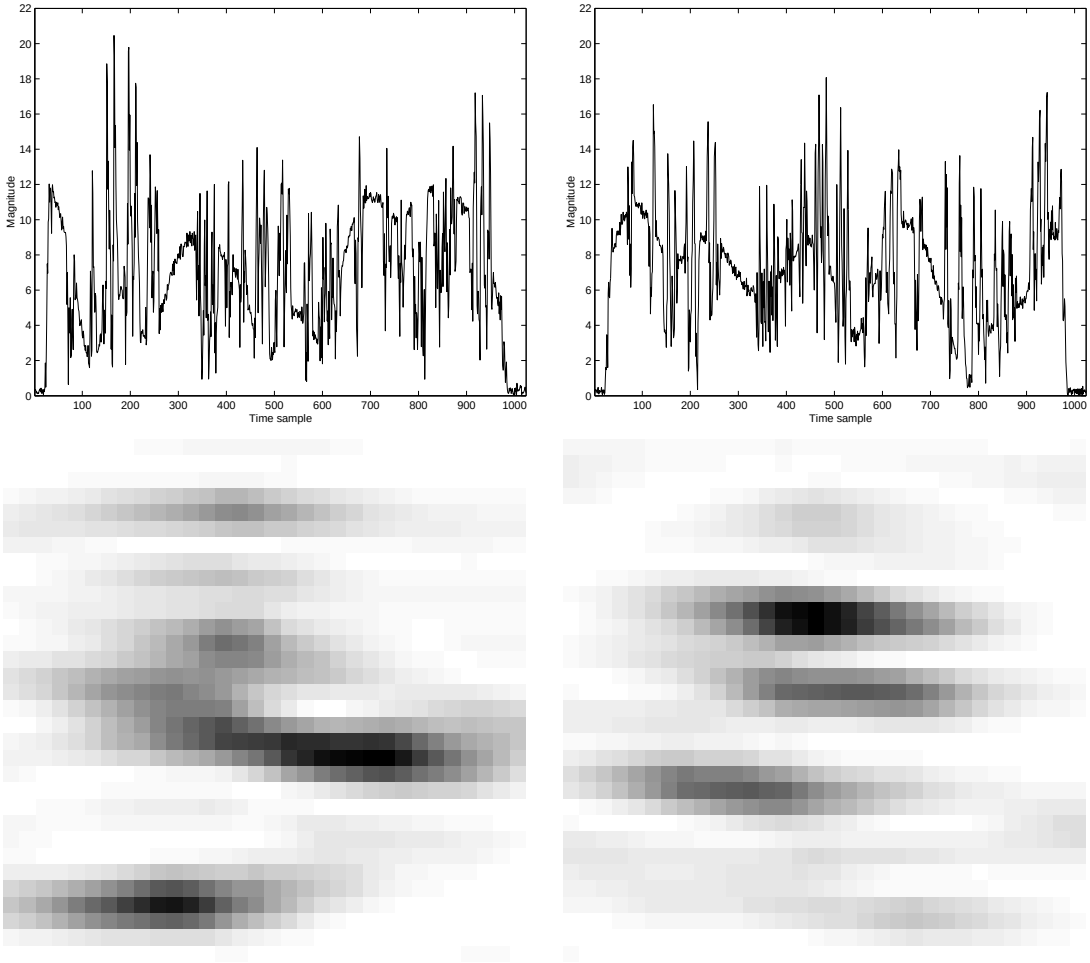


**Figure 2.** Top row: Magnitudes of two data sets resulting from specular realizations of the plane in Fig. 1. Bottom row: Traditional matched filter reconstructions of the delay-doppler image from the data in the top row.

reflectance image, the increased iterations seem to increase the contrast between the object and the background, darkening the outline.

## 6. DIRECTIONS FOR FUTURE WORK

There are numerous ways our implementation could be improved:

- Our prototype implementation was written in MATLAB for ease of debugging and modification. Substantial speedups could be achieved, of course, if it was rewritten in a compiled, lower-level language such as C. Our MATLAB implementation uses the "clear" command frequently to free of space for variables when they are no longer needed; this is necessary to prevent the recursive routines from overflowing the available RAM. This is vital since we have found that the rapid-fire disk accesses associated with virtual memory, colloquially called "thrashing," will essentially bring the algorithm to a grinding halt. If rewritten in a language like C which allows manipulation of pointers, the memory management can be handled at a much more detailed level, enabling larger problems to be addressed.

[§] Falconer[35] refers to matched filtering as imaging with "statistical orthogonality," although matched filtering does not necessarily demand a deep statistical foundation.

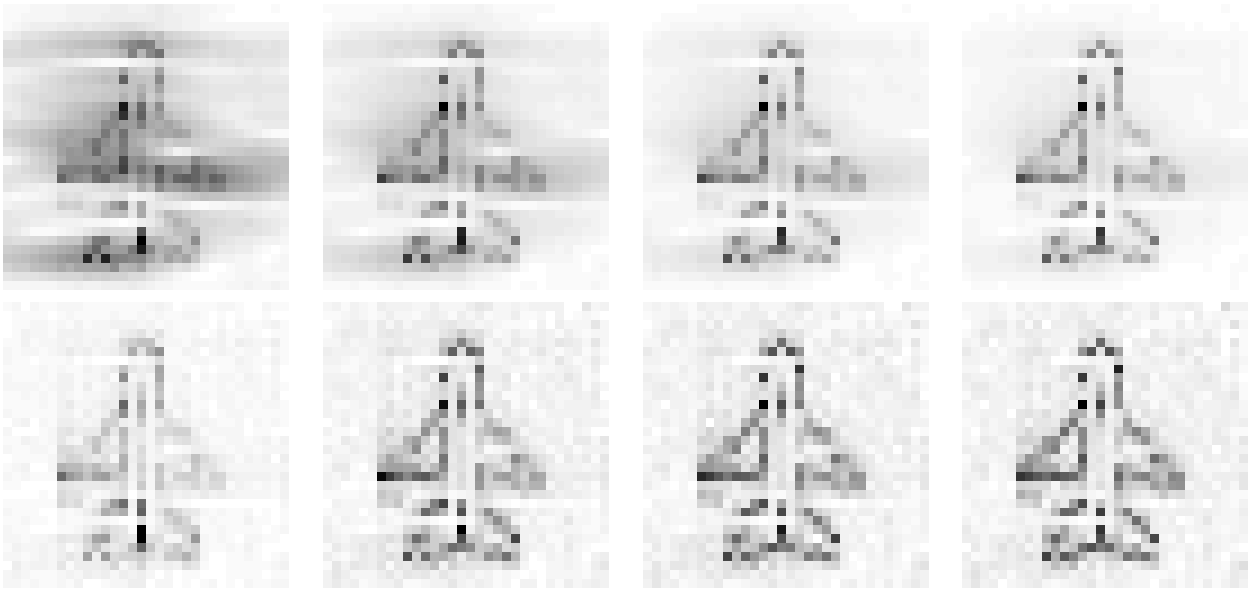

**Figure 3.** Results of first four EM iterations (shown in columns going from left to right) corresponding to the data set in the left panel of Fig. 2. Top row: Scattering function iterations. Bottom row: Reflectance image iterations.

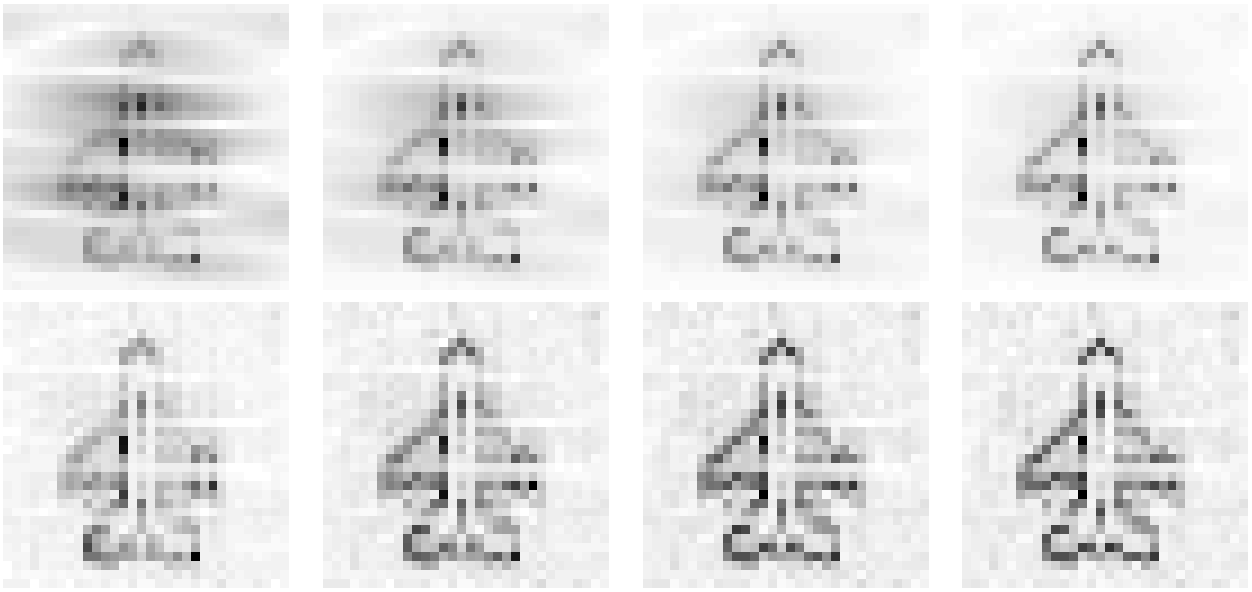

**Figure 4.** Results of first four EM iterations (shown in columns going from left to right) corresponding to the data set in the right panel of Fig. 2. Top row: Scattering function iterations. Bottom row: Reflectance image iterations.

- The recursive algorithms presented above are designed to operate on square matrices, which restricts our implementation to having the length of the data vector equal the number of pixels in the image. Variations of the recursive algorithms have been derived which operate on rectangular matrices.[36–38] These could be used when the number of data points is greater than the number of pixels. (Note that one would generally not want it the other way around).
- Our current implementation of Strassen inversion uses MATLAB's basic "inv" routine for the base case of the recursion, which does not take advantage of the Hermitian symmetry of the matrix to be inverted. Of course,

significant saving could be obtained by exploiting that symmetry, for instance using Cholesky decomposition.

We expect that the EM algorithm for radar imaging could be implemented for large images with run times that are reasonable for interactive research using only modest specialized hardware. For instance, researchers with Apple's Advanced Computation Group boast an implementation of Strassen multiplication which can multiply two real 2500 by 2500 matrices in 40 seconds using the "Velocity Engine" of the G4 processor, which has special instructions for performing vector computations..[32] However, for real-time operation, specialized hardware would probably be required.

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