Signal to Noise

ECE 6279: Spatial Array Processing
Spring 2007
Lecture 13
2/9/07

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Where We Are (and Aren't) in J&D

- Inspired by Section 4.5

Single Plane Wave, Stochastic Model

To keep notation compact, suppress notation of time variable:

Data: $\underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0)\underline{s} + \underline{\mathbf{n}}$

Beamformer output: $z = \mathbf{e}^H(\vec{k})\mathbf{W}\underline{\mathbf{y}}$

$$= \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\underline{s} + \mathbf{e}^H(\vec{k})\mathbf{W}\underline{\mathbf{n}}$$

Signal to Noise Ratio (1)

$$SNR = \frac{E\left[\left|\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\underline{s}\right|^2\right]}{E\left[\left|\mathbf{e}^H(\vec{k})\mathbf{W}\underline{\mathbf{n}}\right|^2\right]}$$

$$= \frac{E\left[\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\underline{s}\underline{s}^H\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})\right]}{E\left[\mathbf{e}^H(\vec{k})\mathbf{W}\underline{\mathbf{n}}\underline{\mathbf{n}}^H\mathbf{W}^H\mathbf{e}(\vec{k})\right]}$$

Signal to Noise Ratio (2)

$$= \frac{\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)E[\underline{s}\underline{s}^H]\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k})\mathbf{W}E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]\mathbf{W}^H\mathbf{e}(\vec{k})}$$

$$= \frac{\sigma_s^2\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{K}_\mathbf{n}\mathbf{W}^H\mathbf{e}(\vec{k})}$$

Special Case of the Numerator (1)

Suppose $\vec{k} = \vec{k}^0$

$$\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0) = \mathbf{e}^H(\vec{k}^0)\mathbf{W}\mathbf{e}(\vec{k}^0) =$$

$$\begin{bmatrix} e^{j\vec{k}^0\cdot\vec{x}_0} & \cdots & e^{j\vec{k}^0\cdot\vec{x}_{M-1}} \end{bmatrix} \begin{bmatrix} w_0 & & \\ & \ddots & \\ & & w_{M-1} \end{bmatrix} \begin{bmatrix} e^{-j\vec{k}^0\cdot\vec{x}_0} \\ \vdots \\ e^{-j\vec{k}^0\cdot\vec{x}_0} \end{bmatrix}$$

$$= \sum_{m=0}^{M-1} w_m$$

Special Case of the Numerator (2)

Suppose $\vec{k} = \vec{k}^0$

$$numer = \sigma_s^2\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})$$

$$= \sigma_s^2\mathbf{e}^H(\vec{k}^0)\mathbf{W}\mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k}^0)$$

$$= \sigma_s^2\left(\sum_{m=0}^{M-1} w_m\right)\left(\sum_{m=0}^{M-1} w_m^*\right) = \sigma_s^2\left|\sum_{m=0}^{M-1} w_m\right|^2$$

Special Case of the Denominator

Special case $\mathbf{K}_\mathbf{n} = \sigma_n^2\mathbf{I}$

$$denom = \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{K}_\mathbf{n}\mathbf{W}^H\mathbf{e}(\vec{k})$$

$$= \sigma_n^2\sum_{m=0}^{M-1}|w_m|^2$$

Both Special Cases

Suppose $\vec{k} = \vec{k}^0$ **and** $\mathbf{K_n} = \sigma_n^2 \mathbf{I}$

$$SNR = \frac{\sigma_s^2 \left| \sum_{m=0}^{M-1} w_m \right|^2}{\sigma_n^2 \left(\sum_{m=1}^{M-1} |w_m|^2 \right)}$$

If $w_m = w_0 : SNR = \dfrac{\sigma_s^2 M^2 w_0^2}{\sigma_n^2 M w_0^2} = \dfrac{\sigma_s^2 M}{\sigma_n^2}$

Two Signal Models

Define the array gain:

$$G = \frac{SNR_{array}}{SNR_{sensor}} = \frac{\sigma_s^2 M / \sigma_n^2}{\sigma_s^2 / \sigma_n^2} = M$$

↑ **in this special case**

Can We Do Better?

- **Schwartz Inequality:**

$$\left| \sum_{m=0}^{M-1} a_m b_m^* \right|^2 \leq \sum_{m=0}^{M-1} |a_m|^2 \sum_{m=0}^{M-1} |b_m|^2$$

with equality iff $a_m = \kappa b_m$

- **Let** $a_m = 1, \; b_m = w_m$

$$\left| \sum_{m=0}^{M-1} w_m \right|^2 \leq M \sum_{m=0}^{M-1} |w_m|^2$$

No, That's the Best We Can Do!

$$\frac{\left| \sum_{m=0}^{M-1} w_m \right|^2}{\sum_{m=0}^{M-1} |w_m|^2} \leq M$$

So in uniform noise, uniform weights maximizes the SNR when beamforming on true wavenumber

A Fresh Approach (1)

To keep notation compact, suppress notation of time variable:

Data: $\underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0)\underline{s} + \underline{\mathbf{n}}$

Beamformer output: $z = \mathbf{a}^H(\vec{k})\underline{\mathbf{y}}$

$$= \mathbf{a}^H(\vec{k})\mathbf{e}(\vec{k}^0)\underline{s} + \mathbf{a}^H(\vec{k})\underline{\mathbf{n}}$$

What choice of $\mathbf{a}$ maximizes SNR?

A Fresh Approach (2)

$$SNR = \frac{\sigma_s^2 \left|\mathbf{a}^H(\vec{k})\mathbf{e}(\vec{k}^0)\right|^2}{\mathbf{a}^H(\vec{k})\mathbf{K}_{\mathbf{n}}\mathbf{a}(\vec{k})}$$

- **Want to study case of a general $\mathbf{K}_{\mathbf{n}}$**
- **Substitute $\tilde{\mathbf{a}} = \mathbf{K}_{\mathbf{n}}^{1/2}\mathbf{a}, \; \mathbf{a} = \mathbf{K}_{\mathbf{n}}^{-1/2}\tilde{\mathbf{a}}$**

$$SNR = \frac{\sigma_s^2 \left|\tilde{\mathbf{a}}^H(\vec{k})\mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^0)\right|^2}{\left|\tilde{\mathbf{a}}(\vec{k})\right|^2}$$

Use the Schwartz!

$$SNR = \frac{\sigma_s^2 \left|\mathbf{a}^H(\vec{k})\mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^0)\right|^2}{\left|\mathbf{a}(\vec{k})\right|^2}$$

- **Schwartz inequality tells us that to maximize SNR, pick**

$$\tilde{\mathbf{a}}(\vec{k}) \propto \mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^0)$$

$$\mathbf{a}(\vec{k}) = \mathbf{K}_{\mathbf{n}}^{-1/2}\tilde{\mathbf{a}} = \mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^0)$$

Beamforming in Colored Noise

$$\mathbf{a}(\vec{k}) = \mathbf{K}_{\mathbf{n}}^{-1}\mathbf{e}(\vec{k}^0)$$

$$z = \mathbf{a}^H(\vec{k})\underline{\mathbf{y}} = \mathbf{e}^H(\vec{k})\mathbf{K}_{\mathbf{n}}^{-1}\underline{\mathbf{y}}$$