

## Signal to Noise

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Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering  
Georgia Institute of Technology  
AL: 404-385-2548  
<lanterma@ece.gatech.edu>

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## Where We Are (and Aren't) in J&D

- Inspired by Section 4.5



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## Single Plane Wave, Stochastic Model

To keep notation compact, suppress notation of time variable:

$$\text{Data: } \underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0) \underline{\mathbf{s}} + \underline{\mathbf{n}}$$

$$\begin{aligned} \text{Beamformer output: } z &= \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{y}} \\ &= \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{\mathbf{s}} + \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \end{aligned}$$

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## Signal to Noise Ratio (1)

$$\begin{aligned} SNR &= \frac{E \left[ \left| \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{\mathbf{s}} \right|^2 \right]}{E \left[ \left| \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \right|^2 \right]} \\ &= \frac{E \left[ \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \underline{\mathbf{s}} \underline{\mathbf{s}}^H \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k}) \right]}{E \left[ \mathbf{e}^H(\vec{k}) \mathbf{W} \underline{\mathbf{n}} \underline{\mathbf{n}}^H \mathbf{W}^H \mathbf{e}(\vec{k}) \right]} \end{aligned}$$

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### Signal to Noise Ratio (2)

$$= \frac{\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) E[\underline{s} \underline{s}^H] \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k}) \mathbf{W} E[\underline{n} \underline{n}^H] \mathbf{W}^H \mathbf{e}(\vec{k})}$$

$$= \frac{\sigma_s^2 \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{K}_n \mathbf{W}^H \mathbf{e}(\vec{k})}$$

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### Special Case of the Numerator (1)

Suppose  $\vec{k} = \vec{k}^0$

$$\mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) = \mathbf{e}^H(\vec{k}^0) \mathbf{W} \mathbf{e}(\vec{k}^0) =$$

$$\begin{bmatrix} e^{j\vec{k}^0 \cdot \vec{x}_0} & \dots & e^{j\vec{k}^0 \cdot \vec{x}_{M-1}} \\ & & \vdots \\ & & w_{M-1} \end{bmatrix} \begin{bmatrix} e^{-j\vec{k}^0 \cdot \vec{x}_0} \\ \vdots \\ e^{-j\vec{k}^0 \cdot \vec{x}_0} \end{bmatrix}$$

$$= \sum_{m=0}^{M-1} w_m$$

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### Special Case of the Numerator (2)

Suppose  $\vec{k} = \vec{k}^0$

$$\text{numer} = \sigma_s^2 \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k})$$

$$= \sigma_s^2 \mathbf{e}^H(\vec{k}^0) \mathbf{W} \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{W}^H \mathbf{e}(\vec{k}^0)$$

$$= \sigma_s^2 \left( \sum_{m=0}^{M-1} w_m \right) \left( \sum_{m=0}^{M-1} w_m^* \right) = \sigma_s^2 \left| \sum_{m=0}^{M-1} w_m \right|^2$$

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### Special Case of the Denominator

Special case  $\mathbf{K}_n = \sigma_n^2 \mathbf{I}$

$$\text{denom} = \mathbf{e}^H(\vec{k}) \mathbf{W} \mathbf{K}_n \mathbf{W}^H \mathbf{e}(\vec{k})$$

$$= \sigma_n^2 \sum_{m=0}^{M-1} |w_m|^2$$

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### Both Special Cases

Suppose  $\vec{k} = \vec{k}^0$  and  $\mathbf{K}_n = \sigma_n^2 \mathbf{I}$

$$SNR = \frac{\sigma_s^2 \left| \sum_{m=0}^{M-1} w_m \right|^2}{\sigma_n^2 \left( \sum_{m=1}^{M-1} |w_m|^2 \right)}$$

If  $w_m = w_0$  :  $SNR = \frac{\sigma_s^2 M^2 w_0^2}{\sigma_n^2 M w_0^2} = \frac{\sigma_s^2 M}{\sigma_n^2}$

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### Two Signal Models

Define the **array gain**:

$$G = \frac{SNR_{array}}{SNR_{sensor}} = \frac{\sigma_s^2 M / \sigma_n^2}{\sigma_s^2 / \sigma_n^2} = M$$

in this special case

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### Can We Do Better?

- **Schwartz Inequality:**

$$\left| \sum_{m=0}^{M-1} a_m b_m^* \right|^2 \leq \sum_{m=0}^{M-1} |a_m|^2 \sum_{m=0}^{M-1} |b_m|^2$$

with equality iff  $a_m = \kappa b_m$

- Let  $a_m = 1$ ,  $b_m = w_m$

$$\left| \sum_{m=0}^{M-1} w_m \right|^2 \leq M \sum_{m=0}^{M-1} |w_m|^2$$

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### No, That's the Best We Can Do!

$$\frac{\left| \sum_{m=0}^{M-1} w_m \right|^2}{\sum_{m=0}^{M-1} |w_m|^2} \leq M$$

So in uniform noise, uniform weights maximizes the SNR when beamforming on true wavenumber

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## A Fresh Approach (1)

To keep notation compact, suppress notation of time variable:

$$\text{Data: } \underline{\mathbf{y}} = \mathbf{e}(\vec{k}^0) \underline{s} + \underline{\mathbf{n}}$$

$$\begin{aligned} \text{Beamformer output: } z &= \mathbf{a}^H(\vec{k}) \underline{\mathbf{y}} \\ &= \mathbf{a}^H(\vec{k}) \mathbf{e}(\vec{k}^0) \underline{s} + \mathbf{a}^H(\vec{k}) \underline{\mathbf{n}} \end{aligned}$$

What choice of  $\mathbf{a}$  maximizes SNR?

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## A Fresh Approach (2)

$$SNR = \frac{\sigma_s^2 \left| \mathbf{a}^H(\vec{k}) \mathbf{e}(\vec{k}^0) \right|^2}{\mathbf{a}^H(\vec{k}) \mathbf{K}_n \mathbf{a}(\vec{k})}$$

- Want to study case of a general  $\mathbf{K}_n$
- Substitute  $\tilde{\mathbf{a}} = \mathbf{K}_n^{1/2} \mathbf{a}$ ,  $\mathbf{a} = \mathbf{K}_n^{-1/2} \tilde{\mathbf{a}}$

$$SNR = \frac{\sigma_s^2 \left| \tilde{\mathbf{a}}^H(\vec{k}) \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0) \right|^2}{\left| \tilde{\mathbf{a}}(\vec{k}) \right|^2}$$

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## Use the Schwartz!

$$SNR = \frac{\sigma_s^2 \left| \mathbf{a}^H(\vec{k}) \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0) \right|^2}{\left| \mathbf{a}(\vec{k}) \right|^2}$$

- Schwartz inequality tells us that to maximize SNR, pick

$$\tilde{\mathbf{a}}(\vec{k}) \propto \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$

$$\mathbf{a}(\vec{k}) = \mathbf{K}_n^{-1/2} \tilde{\mathbf{a}} = \mathbf{K}_n^{-1/2} \mathbf{K}_n^{-1/2} \mathbf{e}(\vec{k}^0)$$

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## Beamforming in Colored Noise

$$\mathbf{a}(\vec{k}) = \mathbf{K}_n^{-1} \mathbf{e}(\vec{k}^0)$$

$$z = \mathbf{a}^H(\vec{k}) \underline{\mathbf{y}} = \mathbf{e}^H(\vec{k}^0) \mathbf{K}_n^{-1} \underline{\mathbf{y}}$$

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