Spatial Averaging and Co-arrays

ECE 6279: Spatial Array Processing
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Lecture 15
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Where We Are in J&D

- **Material from:**
  - –Sec. 4.9.2 on spatial averaging
  - –Sec. 3.3.4 on co-arrays
    - **We will use uniform weights**
  - –Notes by Doug Williams

Why Spatial Averaging?

- **Last lecture, we looked at temporal averaging**
- **Sometimes the source is moving too fast to employ a lot of temporal averaging**
- **If the SNR is still too low, it helps to have some redundancy in the array**

Ex: Uniform Linear Array

- **For a ULA, with multiple uncorrelated sources, and no noise:**

$$\mathbf{R} = \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_{M-1} \\ R_1^* & R_0 & R_1 & \cdots & R_{M-2} \\ R_2^* & R_1^* & R_0 & \cdots & R_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{M-1}^* & R_{M-2}^* & R_{M-3}^* & \cdots & R_0 \end{bmatrix}$$

←Toeplitz structure

Forcing a Toeplitz Structure (1)

$$\hat{\mathbf{R}}_{\mathbf{y}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{y}(l)\mathbf{y}^H(l)$$

$$\hat{\mathbf{R}}_{\mathbf{y}} = \begin{bmatrix} \text{[redacted]} & \text{[redacted]} & \hat{R}_{0,2} & \cdots & \text{[redacted]} \\ \hat{R}^*_{0,1} & \text{[redacted]} & \text{[redacted]} & \cdots & \hat{R}_{1,M-1} \\ \hat{R}^*_{0,2} & \hat{R}^*_{1,2} & \text{[redacted]} & \cdots & \hat{R}_{2,M-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \hat{R}^*_{0,M-1} & \hat{R}^*_{1,M-1} & \hat{R}^*_{2,M-1} & \cdots & \text{[redacted]} \end{bmatrix}$$

$\hat{\hat{R}}_{M-1}$ $\hat{\hat{R}}_1$ $\hat{\hat{R}}_0$

- One idea: Average along diagonals
- Elements near main diagonal get larger variability reduction

Forcing a Toeplitz Structure (2)

- Now we're guaranteed a Toeplitz structure:

$$\hat{\hat{\mathbf{R}}}_{\mathbf{y}} = \begin{bmatrix} \hat{\hat{R}}_0 & \hat{\hat{R}}_1 & \hat{\hat{R}}_2 & \cdots & \hat{\hat{R}}_{M-1} \\ \hat{\hat{R}}^*_1 & \hat{\hat{R}}_0 & \hat{\hat{R}}_1 & \cdots & \hat{\hat{R}}_{M-2} \\ \hat{\hat{R}}^*_2 & \hat{\hat{R}}^*_1 & \hat{\hat{R}}_0 & \cdots & \hat{\hat{R}}_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\hat{R}}^*_{M-1} & \hat{\hat{R}}^*_{M-2} & \hat{\hat{R}}^*_{M-3} & \cdots & \hat{\hat{R}}_0 \end{bmatrix}$$

- …but we're not guaranteed to a nonnegative definite one!
  - i.e, at least one eigenvalue may becomes negative

What Can Go Wrong

- Resulting matrix not guaranteed to be nonnegative definite!
- Also, procedure can introduce some bias into angle estimates
  - Usually only a major problem if correlated signals are present
- One solution: maximum-likelihood structured covariance estimation
  - Expectation-Maximization (EM) algorithm (computationally intensive)
    - M.I. Miller and D.L. Snyder, "The Role of Likelihood and Entropy in Incomplete Data Problems: Application to Estimating Point Process Intensity and Toeplitz Constrained Covariances," Proc. IEEE, Vol. 75, No. 7, pp. 892-907, July 1987.

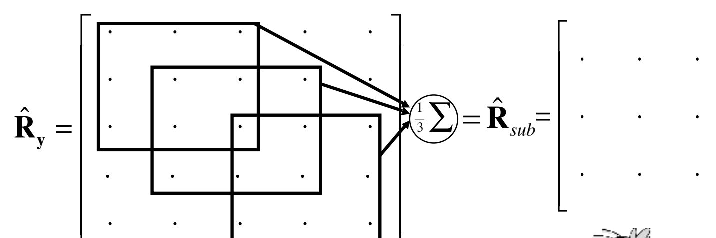
Subaperture Concept

- Suppose we have a 5 element ULA:

○ ○ ○ ○ ○

- Form 3 subarrays:

○ ○ ○ ○ ○
○ ○ ○ ○ ○
○ ○ ○ ○ ○

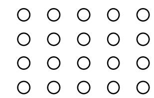


Subaperture Averaging: Pros and Cons

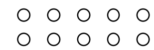
- **Advantages**
  - All elements of estimated spatial covariance matrix (SCM) get the same improvement
  - Resulting SCM estimate is guaranteed to be nonnegative definite
  - Helps reduce problems with correlated sources
    - **Sinusoidal components along diagonals get smoothed out**
- **Disadvantage**
  - Individual subapertures have lower resolution

The Tradeoff

- **Lower variability in covariance estimate comes at the expense of lower resolution**
- **4 small subarrays (bad resolution, low/good variability):**



- **2 big subarrays (good resolution, high/bad variability)**



Forward/Backward Averaging

- **From Prob. 7.11 on p. 418 of J&D**
- **Consider the data for a uniform linear array:**

$$\mathbf{y} = [y_0 \quad y_1 \quad \cdots \quad y_{M-2} \quad y_{M-1}]^T$$

- **Define:**

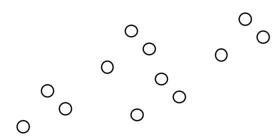
$$\mathbf{y}_r^* = [y_{M-1}^* \quad y_{M-2}^* \quad \cdots \quad y_1^* \quad y_0^*]^T$$

- **Describes same signal and noise situation, but coherence effects differ**

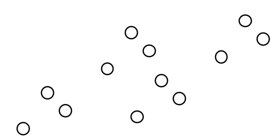
$$\hat{\mathbf{R}}_{fb} = \frac{1}{2}\sum_{l=0}^{L-1}\left\{\mathbf{y}(l)\mathbf{y}^H(l) + \mathbf{y}_r^*(l)[\mathbf{y}_r^*(l)]^H\right\}$$

General Subarrays

- **12 element array**
- **4, 3-element subarrays**



- **2, 6-element subarrays**



Interpreting the SCM

- SCM can be viewed as samples of the correlation function for the entire field
- Samples are taken at differences between sensor locations
- For a single plane wave:

$$[\mathbf{R}]_{m_1,m_2} = \sigma_s^2 \exp\{ jk^0 \cdot (\vec{x}_{m_1} - \vec{x}_{m_2})\}$$

- For a general WSS field :

$$[\mathbf{R}]_{m_1,m_2} = R_f(\vec{x}_{m_1} - \vec{x}_{m_2})$$

where $R_f(\chi) = E[\underline{f}(0)\underline{f}(\chi)]$

The Coarray

- Set of differences between all pairs of sensor locations is called the coarray

$$\bigcup_{m_1,m_2} \{\vec{x}_{m_1} - \vec{x}_{m_2}\}$$

- Differences are known as lags

$$\vec{z}_{m_1,m_2} \equiv \vec{x}_{m_1} - \vec{x}_{m_2}$$

- Elements of the coarray have associated coarray values, which is the number of distinct baselines (pairs of actual sensors) with the same vector difference

Properties of Co-arrays

- Dimension of coarray is same as dimension of array
  - Linear <-> Linear
  - Planar <-> Planar
- Certain lags must exist

$\vec{z}_{m,m} = 0$ (coarray value of *M*)

$$\vec{z}_{m_1,m_2} = -\vec{z}_{m_2,m_1}$$

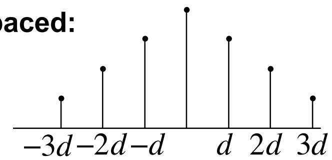
Redundancies

- Repeated lags (i.e. co-array values > 1) are called redundancies
- Zero-lag redundancies occur along main diagonal of SCM
  - Ideally equal if only uncorrelated signals are present
  - Usually not equal due to noise or correlated signals
- Minimum no. of distinct positive lags: M
- Maximum no. of distinct postitive lags: M(M-1)/2 (not achievable if M>4)

Linear Co-array Examples

- **4-sensor uniformly spaced:**

$\circ\; d\; \circ\; d\; \circ\; d\; \circ$



- **4-sensor "perfect" array:**

$\circ\; d\; \circ \quad 3d \quad \circ \quad 2d \quad \circ$