

“Stochastic Signal” Gaussian Model

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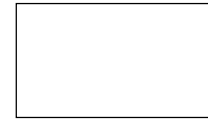
A Stochastic Gaussian Model

- N sources in additive noise

$$\mathbf{y}(l) = \sum_{n=1}^N \mathbf{e}(\boldsymbol{\theta}_n) s_n(l) + \mathbf{w}(l)$$

$$\mathbf{w} \sim CN(0, \mathbf{K}_w)$$

$$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} \sim CN(0, \mathbf{K}_s)$$



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Uncorrelated Sources

- Sources may be uncorrelated (usual model we've used in class)...

$$\mathbf{K}_s = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N^2 \end{bmatrix}$$

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Correlated Sources

- Or they may be correlated (for instance, due to multipath)

$$\mathbf{K}_s = \begin{bmatrix} \sigma_1^2 & c_{12}\sigma_1\sigma_2 & \cdots & c_{1N}\sigma_1\sigma_N \\ c_{12}^*\sigma_1\sigma_2 & \sigma_2^2 & & c_{2N}\sigma_2\sigma_N \\ \vdots & & \ddots & \vdots \\ c_{1N}^*\sigma_1\sigma_N & c_{2N}^*\sigma_2\sigma_N & \cdots & \sigma_N^2 \end{bmatrix}$$

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Fun Fact from Probability

- Independent \Rightarrow uncorrelated
- Uncorrelated $\not\Rightarrow$ independent
(in general)
- Uncorrelated+Gaussian
 \Rightarrow independent



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Distribution of the Data

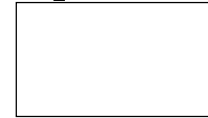
- N sources in additive noise

$$\mathbf{y} \sim CN(0, \mathbf{K}_y)$$

$$\mathbf{K}_y = \mathbf{D}(\Theta) \mathbf{K}_s \mathbf{D}^H(\Theta) + \mathbf{K}_w$$

$$\mathbf{D}(\Theta) = [\mathbf{e}(\theta_1) \cdots \mathbf{e}(\theta_N)]$$

$$\Theta = [\theta_1 \cdots \theta_N]$$



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Probability Density of the Data

$$\mathbf{y} \sim CN(0, \mathbf{K}_y)$$

- Circular (Goodman) zero-mean Gaussian density for L independent snapshots: $p(\mathbf{y}) =$

$$\frac{1}{(\pi^N \det \mathbf{K}_y)^L} \exp \left[- \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right]$$

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Loglikelihood Calculation

- Loglikelihood $\ln p(\mathbf{y}) =$

$$-NL \ln \pi - L \ln \det \mathbf{K}_y - \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l)$$

$$L \operatorname{tr} \left[\frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right]$$



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Trace Rearrangement Trick

$$\begin{aligned} & \text{tr} \left[\frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right] \\ &= \frac{1}{L} \sum_{l=0}^{L-1} \text{tr} \left[\mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right] \\ &= \text{tr} \left[\underbrace{\frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l) \mathbf{K}_y^{-1}}_{\equiv \hat{\mathbf{K}}_y} \right] = \text{tr} \left[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1} \right] \end{aligned}$$

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Maximum-Likelihood Estimation

- **Goal: maximize**
 $-L \ln \det \mathbf{K}_y(\xi) - L \text{tr} \left[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1}(\xi) \right]$
where
 $\mathbf{K}_y(\xi) = \mathbf{D}(\Theta) \mathbf{K}_s \mathbf{D}^H(\Theta) + \mathbf{K}_w$
over ξ , which represents all the parameters (angles, signal powers, correlation coefficients)

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Important Caveat

- The maximization must be over feasible \mathbf{a} , i.e. $\mathbf{K}_y(\xi)$ must be a legitimate covariance matrix (nonnegative definite)
- To make sense at all, we need
 $\dim(\xi) < M^2$

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How Can We Maximize?

- No closed form solution in general
- Still an open research problem!
- One approach: try an **expectation-maximization** algorithm (take ECE7251 to learn about EM algorithms)

M.I. Miller and D.R. Fuhrmann, "Maximum-Likelihood Narrow-Band Direction Finding and the EM Algorithm," IEEE Trans. ASSP, 38(9), Sept. 1990, pp. 1560-1577.

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Adding More Detail

- **Could extend model with:**
 - Unknown noise covariance, for instance

$$\mathbf{K}_w = \mathbf{I} \underbrace{\sigma_w^2}_{\text{Make a parameter}}$$

- Uncertainties in sensor positions

$$x = \tilde{x} + \underbrace{\delta x}_{\text{Make a parameter}}$$

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Alternative: Least Squares Hack

- **Find feasible parameter ξ which minimizes**

$$\left\| \hat{\mathbf{K}}_y - \mathbf{K}_y(\xi) \right\|^2$$

- **Nonlinear Least Squares problem**
 - Large literature on numerical techniques
 - Can use MATLAB Optimization Toolbox
- **Discussed in Sec. 7.1.2 of J&D**

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