

## Special Cases of Maximum-Likelihood Estimation

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## Our Deterministic Gaussian Model

- $N$  sources in additive noise

$$\mathbf{y}(l) = \sum_{n=1}^N \mathbf{e}(\boldsymbol{\theta}_n) s_n(l) + \mathbf{w}(l)$$

$$\mathbf{w} \sim CN(0, \mathbf{K}_w)$$

$$\mathbf{s}(l) = \begin{bmatrix} s_1(l) \\ \vdots \\ s_N(l) \end{bmatrix}$$

Modeled as deterministic parameters we need to estimate

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## The Single-Source Case

- Recall ML estimate of signal for a fixed  $\Theta$

$$\hat{\mathbf{s}} = [\mathbf{D}^H(\Theta) \mathbf{K}_w^{-1} \mathbf{D}(\Theta)]^{-1} \mathbf{D}(\Theta)^H \mathbf{K}_w^{-1} \mathbf{y}$$

- Suppose we only have one source

$$\hat{s} = [\mathbf{e}^H(\theta) \mathbf{K}_w^{-1} \mathbf{e}(\theta)]^{-1} \mathbf{e}(\theta)^H \mathbf{K}_w^{-1} \mathbf{y}$$

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## A Blast from the Past

$$\hat{s} = \frac{\mathbf{e}^H(\theta) \mathbf{K}_w^{-1} \mathbf{y}}{\underbrace{\mathbf{e}^H(\theta) \mathbf{K}_w^{-1} \mathbf{e}(\theta)}_{\equiv \mathbf{a}^H}} = \mathbf{a}^H \mathbf{y}$$

- Recall  $\mathbf{a}^H \propto \mathbf{e}^H(\theta) \mathbf{K}_w^{-1}$  are the weights that maximize SNR

–Showed this  
in Lecture 13

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## Another Blast from the Past

- If  $\mathbf{K}_w = \sigma_w^2 \mathbf{I}$  then

$$\hat{\mathbf{s}} = \frac{\mathbf{e}^H(\boldsymbol{\theta}) \mathbf{K}_w^{-1}}{\mathbf{e}^H(\boldsymbol{\theta}) \mathbf{K}_w^{-1} \mathbf{e}(\boldsymbol{\theta})} \mathbf{y} = \frac{\mathbf{e}^H(\boldsymbol{\theta})}{\mathbf{e}^H(\boldsymbol{\theta}) \mathbf{e}(\boldsymbol{\theta})} \mathbf{y}$$

$$= \frac{\mathbf{e}^H(\boldsymbol{\theta})}{M} \mathbf{y}$$

- Old delay-and-sum conventional beamformer

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## Maximum-Likelihood Procedure

- ML procedure said to maximize  $\text{tr}\{\mathbf{D}(\mathbf{D}^H \mathbf{D})^{-1} \mathbf{D}^H \hat{\mathbf{R}}_y\}$  over  $\Theta$

- For one source

$$\text{tr}\{\mathbf{e}(\mathbf{e}^H \mathbf{e})^{-1} \mathbf{e}^H \hat{\mathbf{R}}_y\} = \frac{1}{M} \text{tr}\{\mathbf{e} \mathbf{e}^H \hat{\mathbf{R}}_y\}$$

$$= \frac{1}{M} \text{tr}\{\mathbf{e}^H \hat{\mathbf{R}}_y \mathbf{e}\} = \underbrace{\mathbf{e}^H(\boldsymbol{\theta}) \hat{\mathbf{R}}_y \mathbf{e}(\boldsymbol{\theta})}_{\text{Power out of our old conventional delay-and-sum beamformer!}} / M$$

Power out of our old conventional delay-and-sum beamformer!

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## Take-Home Messages (1)

- If you have a **single source in white Gaussian noise**, maximizing the SNR turns out to be optimal (in the ML estimation sense) under the deterministic Gaussian model
- If you have non-Gaussian noise or multiple sources, maximizing SNR is **not** the optimal ML procedure
  - But usually still pretty useful!

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## Take-Home Message (2)

- If you have a **single source in white Gaussian noise**, the conventional delay-and-sum beamformer is optimal in the ML sense under the deterministic Gaussian model
- If you have non-Gaussian noise or multiple sources the conventional beamformer is **not** the optimal ML procedure
  - But usually still pretty useful!

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### Capon's Argument (1)

- ML signal estimate for one source is 
$$\hat{\mathbf{s}} = \frac{\mathbf{e}^H(\boldsymbol{\theta})\mathbf{K}_w^{-1}}{\mathbf{e}^H(\boldsymbol{\theta})\mathbf{K}_w^{-1}\mathbf{e}(\boldsymbol{\theta})}\mathbf{y}$$
- In Capon's application (see p. 358-359 of J&D), he didn't have a good estimate of  $\mathbf{K}_w$
- Capon handwaved and said to try replacing  $\mathbf{K}_w$  with  $\hat{\mathbf{R}}_y$

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### Capon's Argument (2)

- Totally ad-hoc from an ML standpoint;  $\hat{\mathbf{R}}_y$  is a terrible estimate of  $\mathbf{K}_w$
- But if we go ahead and do it, we get 
$$\hat{\mathbf{s}} = \frac{\mathbf{e}^H(\boldsymbol{\theta})\hat{\mathbf{R}}_y^{-1}}{\mathbf{e}^H(\boldsymbol{\theta})\hat{\mathbf{R}}_y^{-1}\mathbf{e}(\boldsymbol{\theta})}\mathbf{y}$$
 which is the MVDR beamformer!

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### A Great Confusion

- MVDR (a.k.a. Capon's method) is sometimes erroneously referred to as maximum-likelihood
  - That's nonsense! MVDR does not arise from any known statistical ML procedure
- Our earlier derivation of MVDR from a constrained optimization procedure is due to Lacoss (1971)

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### Recollections and Observations

- Remember MVDR was a solution to 
$$\mathbf{a}^H \mathbf{R}_y \mathbf{a} \text{ s.t. } \mathbf{a}^H \mathbf{e}(\boldsymbol{\theta}) = 1$$
- Yes, this is an optimization problem; but it's not a maximum-likelihood optimization problem
  - Doesn't require a probability model for the data

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## Our Stochastic Gaussian Model

- $N$  sources in additive noise

$$\mathbf{y}(l) = \sum_{n=1}^N \mathbf{e}(\boldsymbol{\theta}_n) s_n(l) + \mathbf{w}(l)$$

$$\mathbf{w} \sim CN(0, \mathbf{K}_w)$$

$$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} \sim CN(0, \mathbf{K}_s)$$

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## Maximum-Likelihood Estimation

- Goal: maximize

$$-L \ln \det \mathbf{K}_y(\boldsymbol{\xi}) - L \operatorname{tr}[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1}(\boldsymbol{\xi})]$$

where

$$\mathbf{K}_y(\boldsymbol{\xi}) = \mathbf{D}(\boldsymbol{\Theta}) \mathbf{K}_s \mathbf{D}^H(\boldsymbol{\Theta}) + \mathbf{K}_w$$

over  $\boldsymbol{\xi}$ , which represents all the parameters (angles, signal powers, correlation coefficients)

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## Asymptotic ML Est. for Stochastic Model

- If you have
  - “Lots of snapshots,” and
  - Uncorrelated sources
  - I.I.D. receiver noise
- Then VT-IV, pp. 985-991 shows that you can get rid of the power parameters  $\sigma_1^2, \dots, \sigma_N^2$  and find ML estimates of angles by maximizing...

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## Asymptotic ML Procedure

- New goal: maximize

$$-\ln \det[\mathbf{P}_D \hat{\mathbf{K}}_y \mathbf{P}_D + \sigma_w^2 \mathbf{P}_D^\perp] - \operatorname{tr}[\mathbf{P}_D^\perp \hat{\mathbf{K}}_y] / \sigma_w^2$$

over  $\boldsymbol{\Theta}$

$$\text{where } \mathbf{P}_D = \mathbf{D}(\mathbf{D}^H \mathbf{D})^{-1} \mathbf{D}^H$$

$$\mathbf{P}_D^\perp = \mathbf{I} - \mathbf{P}_D$$

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