

## Delay-and-Sum Beamforming for Spherical Waves

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## Where We Are in J&D

- Lecture material drawn from:
  - Sec. 4.1.3
  - Sec. 4.2.1 (“Point Focusing” part on p. 123)
- Please read section 4.1.1 on your own

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## Delay-and-Sum Beamforming

- Array of  $M$  sensors at positions  $\vec{x}_0 \dots \vec{x}_{M-1}$
- For convenience, put the phase center at the origin

$$\frac{1}{M} \sum_{m=0}^{M-1} \vec{x}_m = \vec{0}$$

- Delay-and-sum beamforming

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

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## Spherical Waves at the Sensors

Source location

$$f(\vec{x}, t) = \frac{s(t - |\vec{x} - \vec{x}^0|/c)}{|\vec{x} - \vec{x}^0|}$$

$$y_m(t) = \frac{s(t - |\vec{x}_m - \vec{x}^0|)}{|\vec{x}_m - \vec{x}^0|} = \frac{s(t - r_m^0/c)}{r_m^0}$$

where  $r_m^0 = |\vec{x}_m - \vec{x}^0|$  ← Distance between source and sensor  $m$

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## Delay-and-Sum for Spherical Waves

$$y_m(t) = \frac{s(t - r_m^0/c)}{r_m^0}$$

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

$$= \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0/c)}{r_m^0}$$

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## Matched Delays (1)

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0/c)}{r_m^0}$$

- If we pick  $\Delta_m = (r^0 - r_m^0)/c$  then we get

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - r^0/c)}{r_m^0}$$

$$= \frac{s(t - r^0/c)}{r^0} \left[ \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \right]$$

Distance between source and origin

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## Matched Delays (2)

$$z(t) = \underbrace{\frac{s(t - r^0/c)}{r^0}}_{\text{Original signal, delayed and attenuated}} \left[ \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} \right]_{\text{Approaches sum of weights as } r^0, r_m^0 \rightarrow \infty}$$

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## Mismatched Delays

$$z(t) = \sum_{m=0}^{M-1} w_m \frac{s(t - \Delta_m - r_m^0/c)}{r_m^0}$$

Distance between assumed source and origin      Distance between assumed source and sensor  $m$

- In general, if we pick  $\Delta_m = (r - r_m^0)/c$  then we get

$$z(t) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} s\left(t - \frac{r - (r_m^0 - r^0)}{c}\right)$$

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### Beamforming Strategy

$$\Delta_m = (r - r_m) / c$$

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

$$= \sum_{m=0}^{M-1} w_m y_m\left(t - \frac{r - r_m}{c}\right)$$

- Find the  $(x)$  which gives the most energy in  $z(t)$



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### Time Signal Seen at the Origin

$$s_0(t) = \frac{s(t - r^0 / c)}{r^0}$$

$$S_0(\omega) = \frac{1}{r^0} S(\omega) \exp(-j\omega r^0 / c)$$

$$r^0 S_0(\omega) \exp(j\omega r^0 / c) = S(\omega)$$



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### Output in Fourier Domain (1)

$$z(t) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} s\left(t - \frac{r - (r_m - r_m^0)}{c}\right)$$

Take FT of both sides

$$Z(\omega) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} S(\omega) \times \exp\left[-j\omega \frac{r - (r_m - r_m^0)}{c}\right]$$



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### Output in Fourier Domain (2)

$$Z(\omega) = \sum_{m=0}^{M-1} \frac{w_m}{r_m^0} S(\omega) \exp\left[-j\omega \frac{r - (r_m - r_m^0)}{c}\right]$$

$$S(\omega) = r^0 S_0(\omega) \exp(j\omega r^0 / c)$$

$$Z(\omega) = \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m^0} S_0(\omega) \times$$

$$\exp\left[j\omega \left\{ \frac{(r^0 - r) - (r_m^0 - r_m)}{c} \right\}\right]$$

$$k = \omega / c$$



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### Array-Pattern-Like Thing

$$Z(\omega) = S_0(\omega) \widehat{W}(k, \vec{x}, \vec{x}^0)$$

Where  $\widehat{W}(k, \vec{x}, \vec{x}^0) \equiv$   $k = \omega/c$

$$\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m} \exp[jk\{(r^0 - r) - (r_m^0 - r_m)\}]$$

$\widehat{W}(k, \vec{x}, \vec{x}^0)$  plays a role analogous to  $W(\vec{\omega}^0 \vec{\alpha} - \vec{k}^0)$  in previous lecture on beamforming for plane waves

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### When Perfectly Focused

- If  $\vec{x} = \vec{x}^0$ , i.e.,  $r^0 = r$  and  $r_m^0 = r_m$

$$\widehat{W}(k, \vec{x}, \vec{x}^0) \equiv$$

$$\sum_{m=0}^{M-1} w_m \frac{r^0}{r_m} \exp[jk\{(r^0 - r^0) - (r_m^0 - r_m^0)\}]$$

$$= \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m} \longrightarrow z(t) = s_0(t) \left[ \sum_{m=0}^{M-1} w_m \frac{r^0}{r_m} \right]$$

**We copy the signal exactly - no filtering!**

Now show movie

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