Pisarenko Harmonic Decomposition

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Lecture 18

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Where We Are in J&D

- **Section 7.2.5**

Previous "Minimum Variance" Techniques

- **General linear constraint**

$$\mathbf{w}_{\langle\rangle} = \arg\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R}_{\mathbf{y}} \mathbf{w} \text{ s.t. } \mathbf{C}\mathbf{w} = \mathbf{c}$$

- **"Distortionless Response" constraint**

$$\mathbf{w}_{\langle\rangle} = \arg\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R}_{\mathbf{y}} \mathbf{w} \text{ s.t. } \mathbf{e}^H(\vec{k})\mathbf{w} = 1$$

New "Minimum Variance" Technique

- **Power constraint on weights**

$$\mathbf{w}_{\langle\rangle} = \arg\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R}_{\mathbf{y}} \mathbf{w} \text{ s.t. } \mathbf{w}^H \mathbf{w} = 1$$

- **Philosophy: To find where signals are, look for where they aren't**

The Lagrangian

- Power constraint on weights

$$L(\mathbf{w},\boldsymbol{\lambda}) = \mathbf{w}^H \mathbf{R}_{\mathrm{y}} \mathbf{w} + \lambda(\mathbf{w}^H \mathbf{w} - 1)$$

$$\nabla_{\mathbf{w}^*} L(\mathbf{w},\boldsymbol{\lambda}) = \mathbf{R}_{\mathrm{y}} \mathbf{w} + \boldsymbol{\lambda} \mathbf{w} = 0$$

$$\mathbf{R}_{\mathrm{y}} \mathbf{w} = -\lambda \mathbf{w}$$

- $\mathbf{w}$ must be an eigenvector of $\mathbf{R}_{\mathrm{y}}$

Eigenvector/Eigenvalue Analysis

$$\mathbf{w}^H \mathbf{R}_{\mathrm{y}} \mathbf{w} = \sum_{m=1}^{M} \lambda_m \left| \mathbf{w}^H \mathbf{v}_m \right|^2$$

- To minimize $\mathbf{w}^H \mathbf{R}_{\mathrm{y}} \mathbf{w}$, pick the eigenvector $\mathbf{v}_{\min}$ corresponding to the smallest eigenvalue of $\mathbf{R}_{\mathrm{y}}$

$$\mathbf{w} = \mathbf{v}_{\min}$$

- In practice, must use an estimate $\hat{\mathbf{R}}_{\mathrm{y}}$

Strategy

$\left| \mathbf{v}_{\min}^H \mathbf{e}(\vec{k}) \right|$ is small when $\vec{k}$ corresponds to an actual signal

$\dfrac{1}{\left| \mathbf{v}_{\min}^H \mathbf{e}(\vec{k}) \right|}$ is large when $\vec{k}$ corresponds to an actual signal

Warnings

- Spurious peaks
- Peak values have little to do with actual signal amplitudes
- Resolution depends on quality of estimate of $\mathbf{R}_{\mathrm{y}}$
- Helps to know the number of targets ahead of time

Rank One Approximations

$$\mathbf{R_y} = \sum_{m=1}^{M} \lambda_m \mathbf{v}_m \mathbf{v}_m^H \approx \lambda_{\max} \mathbf{v}_{\max} \mathbf{v}_{\max}^H$$

$$\mathbf{R_y}^{-1} = \sum_{m=1}^{M} \frac{1}{\lambda_m} \mathbf{v}_m \mathbf{v}_m^H \approx \frac{1}{\lambda_{\min}} \mathbf{v}_{\min} \mathbf{v}_{\min}^H$$

Comparisons and Interpretations

$$P^{CONV}(\vec{k}) \equiv \mathbf{e}^H(\vec{k})\mathbf{R_y}\mathbf{e}(\vec{k})$$

$$P^{MV}(\vec{k}) \equiv \left[\mathbf{e}^H(\vec{k})\mathbf{R_y}^{-1}\mathbf{e}(\vec{k})\right]^{-1}$$

$$P^{P}(\vec{k}) \equiv \left|\mathbf{v}_{\min}^H \mathbf{e}(\vec{k})\right|^{-2}$$

$$= \left[\mathbf{e}^H(\vec{k})\underbrace{\mathbf{v}_{\min}\mathbf{v}_{\min}^H}\mathbf{e}(\vec{k})\right]^{-1}$$

proportional to a rank-one approximation of $\mathbf{R_y}^{-1}$